

An 'Alive' DGS Tool for Students' Cognitive Development

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Abstract— A basic goal of the current study, which is an excerpt from a larger study, is to analyse students' interactions in the context of their working on transformations of tools, and specifically of custom tools in a microworld, the Geometer's Sketchpad. Custom tools are encapsulated objects created in a DGS environment. The construction of a custom tool and its subsequent implementation in a pair of students are the focus of this study. Custom tools can serve as structural units of knowledge, as conceptual objects and hence as 'schemes'. Moreover, they can become an 'alive' active tool for students' cognitive development. The paper will include the following parts: (a) how students learn in a constructivist framework; (b) a description of the van Hiele model, and especially the meanings of 'symbol and signal character'; (c) how a DGS environment functions as an 'alive' microworld; (d) the role of artifacts-[custom] tools as instruments-[custom] tools; (e) the research methodology of the current study (f) a detailed description of the experimental process (g) discussion and conclusion.

Keywords— dynamic geometry environment; 'alive' tool; custom tool; cognitive development

I. INTRODUCTION

Researchers, educators and teachers of mathematics around the world agree that learning mathematics is a complex process, as we understand it in terms of both constructivist (:an individual sense-making learning theory) and sociocultural (:a theory that supports learning through social interactions) theories.

Constructivist learning builds on the students' pre-existing knowledge. Bransford, Brown & Cocking (2000) support that "constructivists assume that all knowledge is constructed from previous knowledge, irrespective of how one is taught (e.g., Cobb, 1994)—even listening to a lecture involves active attempts to construct new knowledge" (p. 11). They point out that "Like 'Fish is Fish' everything the children hear [is] incorporated into [their] pre-existing view". 'Fish is Fish' (Lionni, 1970, cited in Bransford et al., 2000) is a tale in which a fish tries to understand how people and cows appear/exist in the external world from the descriptions of a frog that has gone outside to view everything.

"The book shows pictures of the fish's representations of each of these descriptions: each is a fish-like form that is slightly adapted to accommodate the frog's descriptions. [...] This tale illustrates both the creative opportunities and dangers inherent in the fact that people construct new knowledge based on their current knowledge." (Bransford et al., 2000, p. 11).

Representations may be considered "as internal—abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience" (Pape, & Tchoshanov, 2001, p. 119). In the words of Piaget (1970), a scheme refers to "whatever is repeatable and generalizable in an action" (p.42). Vergnaud (2009) also supports that "representation is a dynamic activity [...] and] schemes are essential: they organize gestures and action in the physical world, as well as interaction with others, conversation, and reasoning" (p.93).

The activity of solving problems is based on the interaction and transformation between different representational systems (e.g., Goldin & Janvier, 1998) of the same meaning. The ability to interpret a meaning between representational systems (Janvier, 1987) is necessary for students' conceptual understanding in mathematics.

A computer microworld is an external representational system. One of the basic goals of the current study, which is an excerpt from a larger study, is to analyse students' interactions in the context of their working on transformations of tools, and specifically of custom tools in a microworld, the Geometer's Sketchpad (Jackiw, 1991/ 2001). According to Sketchpad Help System: "Custom tools are tools that you create yourself [...]. By defining new custom tools, you extend the built-in tools available to you in Sketchpad". The construction of a custom tool and its subsequent implementation are the focus of this study. Moreover, *instrumental decoding* is the competency of the student to transform his/her mental

representations to actions on screen by using the software's tools (e.g., Patsiomitou, 2011, 2012a, b).

As Vico (1710) argued "we can know nothing that we have not made" (cited in Ernest, 1994, p.76). In Ernest's (1994) opinion "microcomputers have great potential [...], because they encourage children to think 'outside their heads', providing direct evidence of children's learning and thought processes" (p.21).

A teacher also can construct a tool enhancing his/her student's understanding and anticipating their needs during an investigation in a problem-solving situation. In this way, the tool mediates between the teacher/instructor's thought and the students' learning of mathematics and becomes a Vygotskian 'psychological tool'. A custom tool is a digital tool that becomes an 'instrument' (i.e. a Vygotskian 'sign') during *instrumental genesis* process, involving the construction of personal schemes or, more generally, "*the appropriation of social pre-existing schemes*" (Artigue, 2000, p.10). Beguin & Rabardel (2000) support that an 'instrument' is the evolution of a tool (material or semiotic) only when a student is able to construct *utilization schemes of its use* and can manage this tool in an activity. Then the student is able to construct *schemes of instrumented action*, which progressively are incorporated into software's techniques, allowing him/her to solve given tasks efficiently (Artigue, 2000). I recognised in van Hiele's theory a proper frame for the description of student's levels and concretely the students gradually acquisition of a figure's *symbol character* and then of a figure's *signal character* (van Hiele, 1986; Choi-Koh, 1999) *as well as the competency to reverse their thinking* (e.g., Patsiomitou, 2012a, b). The current study will relate in particular to the description and analysis of: (a) interactions during the utilization of the DGS tools' transformations and where the schemes of instrumented action are identified and performed; (b) the impact of interactions on students participating in the same group and the impact of researchers' interventions on the students' construction of meanings; (c) students' verbal formulations and their modifications as they occur in interaction with the dynamic diagram. With these in mind, I investigated the following questions:

- (a) What is the effect of a custom tool on a student's construction of instrumented action schemes and, subsequently, on their conceptual understanding and construction of meanings?
- (b) What kinds of transformations take place during the experimental process?
- (c) What are the concepts developed during the process in correlation with the utilization of the tools?
- (d) Does the students' instrumental decoding of the tools affect their cognitive development?

The paper will include the following parts: (a) how students learn in a constructivist framework; (b) a description of the van Hiele model, and especially the meanings of 'symbol and signal character'; (c) how a DGS software package functions as an 'alive' microworld; (d) the role of artifacts-[custom] tools as instruments-[custom] tools; (e) the research methodology of the current study (f) a detailed description of

the experimental process (g) discussion and conclusion(/sumperasmata).

II. HOW DO STUDENTS LEARN IN A CONSTRUCTIVIST FRAMEWORK?

A constructivist view of learning considers the student as an *active participant* and learning as an *active process*. Immanuel Kant (1965), John Dewey (e.g., 1938/1988), Jean Piaget (e.g., 1937/1971, 1970), von Glasersfeld (e.g., 1995), Vygotsky (e.g., 1934/1962, 1978) and Skemp (e.g., 1987) were important philosophers and theorists who gradually changed the traditional "*route by memorization*", the behaviourists' view of learning mathematics, to a sociocultural-constructivist view of learning mathematics. From an epistemological point of view, constructivism emphasizes the construction of meanings in collaboration between the instructor (or /teacher-action researcher) and the student (e.g., Hayes & Oppenheim, 1997). According to O'Toole and Plummer (2004)

"Taking the view that mathematics is not static but rather humanistic field that is continually growing and reforming, and that children construct their own knowledge (Hersch, 1997), then teaching can no longer be a matter of viewing students' minds as 'empty vessels' ready to adopt internalise and reproduce correct mathematical knowledge and applications. Rather, we have come to learn that teaching which includes instructional contexts where students are supported to move from their own intuitive mathematical understandings to those of conventional mathematics, produces more profound levels of mathematical understandings (Skemp, 1971)" (p. 3).

Piaget (1937/1971) considered that students' thinking becomes more sophisticated with biological maturity. Students build on their own intellectual structures as they grow up. He introduced the development of student's thinking in stages, based on the process of equilibration and the mechanisms of assimilation and accommodation. In the last chapter of his work "The Construction of Reality in the Child" translated by M. Cook, Piaget (1937/1971) stated that:

"[...] In their initial directions, assimilation and accommodation are obviously opposed to one another, since assimilation is conservative and tends to subordinate the environment to the organism as it is, whereas accommodation is the source of changes and bends the organism to the successive constraints of the environment [...] Assimilation and accommodation are therefore the two poles of an interaction between the organism and the environment, which is the condition for all biological and intellectual operation, and such an interaction presupposes from the point of departure an equilibrium between the two tendencies of opposite poles."(pp.2-3)

In other words, Piaget supports that students construct new concepts, 'assimilating' in a conservative way or 'accommodating' in a modifying way their prior knowledge conceptions. In a constructivist approach the reference to

schemes is essential. Littlefield-Cook, & Cook (2005) support that

“For Piaget, the essential building block for cognition is the *scheme*. A scheme is an organized pattern of action or thought. It is a broad concept and can refer to organized patterns of physical action (such as an infant reaching to grasp an object), or mental action (such as a high school student thinking about how to solve an algebra problem). As children interact with the environment, individual schemes become modified, combined, and reorganized to form more complex cognitive structures” (p.6).

Since tools exert an influence over the technical and social way in which students conduct an activity, they are considered essential to their cognitive development. According to Vygotsky (1978), tools can be considered as external signs and they can become tools of semiotic mediation. He developed the zone of proximal development and defined it as “the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers” (p.86).

In Vygotsky's theory, it is taken for granted that less advanced students can learn from their peers who have more competence to solve problems and can interpret a meaning between representational systems. Vygotsky also argues that “if learning can be influenced by social mediation, then conditions can be created in schools than can help students learn” (Vygotsky, 1978 p. 86). The language development is a central idea in the theory of Vygotsky, something that is also common to the theory of van Hiele (Fuys et al., 1984, 1988). Moreover, the mathematical social discourses developed in a small group mediated by cognitive tools enhance the social interactions in class and support the development of students' mathematical communication and understanding of mathematical concepts.

As Littlefield-Cook, & Cook (2005) support “it is the language that carries the concepts and cognitive structures to the child, and these concepts become the “*psychological tools*” that the child will use (Vygotsky, 1962)” (p. 26). This is in accordance with the view that learning is an ongoing and evolving importance for students' language development, as well as their development of mathematical terminology and conceptual understanding. Moreover, in the words of Sfard (2001) “[...] we can define learning as the process of changing one's discursive ways in a certain well-defined manner.” (p. 3).

Steffe & Tzur (1994) in their article “Interaction and Children's mathematics” argued that learning “occurs as a product of interaction [and] the teacher's interventions is essential in children's learning. But in this, we speak in terms of perturbations as well as in terms of provocations, because it is the children who must experience the perturbations” (p. 44). Many teachers try to apply a learning theory's principles to their instruction (though they do not usually achieve the expected results). Others try a combination of theories: drill

and practice (a behaviourist view of learning), enquiry and constructivist learning using ICT – in other words, a multiple-theories approach whose results depend on the teacher's different types of knowledge [based on Schulman (1987) and Mishra and Koehler's (2006) framework of Technology, Pedagogy, and Content Knowledge (TPACK)], the students' backgrounds, external resources in the school environment, etc. Critics of the multiple-theories approach to teaching argue that moving back and forth between theories of learning reduces (or eliminates) the coherence, insights and results provided by a single theory, even if this interplay is between theories with complementary perspectives, such as constructivist and sociocultural theories (e.g., Confrey, 1995; Lerman, 1996). Bransford, Brown & Cocking (2000, p. 22) created an image (Fig. 1) in which they present “how people learn, which teachers can choose more purposefully among techniques to accomplish specific goals”.

In my opinion, student learning does not work as a machine into which data, information and the principles of a learning theory are entered and the expected results come out. On the other hand, is the merging of *constructivist and sociocultural perspectives* a theory we can apply to instructional processes and the everyday teaching of mathematics? Can we construct learning paths to apply the principles of constructivism to student's learning? As Fosnot (2003) states

“Although educators now commonly talk about a “constructivist-based” practice as if there is such a thing, in reality constructivism is not a theory of teaching; *it is a theory about learning*. In fact, as we shift our teaching towards trying to support cognitive construction, the field of education has been left without well-articulated theories of teaching. [...] Major questions loom around what should be taught, how we should teach, and how best to educate teachers for this paradigmatic shift. The problem is that all of these pedagogical strategies can be used without the desired learning resulting. This is because constructivism is a theory of learning, not a theory of teaching, and many educators who attempt to use such pedagogical strategies confuse discovery learning and “hands-on” approaches with constructivism”.

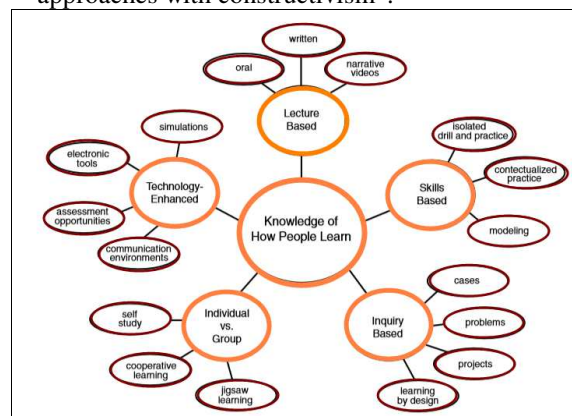


Fig. 1. “Knowledge of how people learn” (Bransford, Brown & Cocking, 2000, p.22), adapted

Bruner (1966) developed an instructional theory. Bruner emphasized the teacher's proper use of language when s/he introduces a meaning to children. Discovery learning was also advocated by Bruner (1961, 1966). He pointed out that discovery learning "increases the interest of students, creates exciting classroom atmosphere, encourages and increases participation, provokes enthusiasm and inquiry, and helps students learn new content" (Bayram, 2004, p.40). Within the theory developed by Bruner (1966) *cognitive conflict* "occurs when there is a mismatch between information encoded in two of the representational systems, between [...] what one sees and how one says it [...]" (Bruner, Olver, & Greenfield, 1966, p. 11). If the student overcomes this contradiction he is able to mental growth. In a constructivist frame, cognitive conflict is a basic component in the learning process (Karmiloff-Smith & Inhelder, 1974) and very important for the development of students' geometrical thinking. Van Hiele also developed a theoretical model for thought development that can be applied to students' instruction. I shall present their model in the next section.

III. THE VAN HIELE MODEL: SYMBOL AND SIGNAL CHARACTER

Pierre van Hiele and his wife Dina van Hiele-Geldof developed a theoretical model of thought development in geometry. The van Hieles distinguished five different levels of thought and how the students progress through levels, during the instruction. Dina van Hiele-Geldof (1957/1984) in her didactic experiments investigated *"the improvement of learning performance by a change in the learning method"* (p.16). She investigated whether it was possible to use instruction as a way of presenting material to participated students, so that the holistic visual thinking of a child can be transformed into concrete abstract thinking in a continuous process, something that is prerequisite for the development of deductive reasoning in geometry.

"After observing secondary school' students having great difficulty learning geometry in their classes, Dutch educators Pierre van Hiele and his wife, Dina van Hiele-Geldof developed a theoretical model involving five levels of thought development in geometry. Their work, which focuses' on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next, was first reported in companion dissertations at the University of Utrecht in 1957." (Fuys et al., 1984, p.6).

Pierre van Hiele finally, characterized his model in terms of three rather than five levels of thought: visual (level 1), descriptive (level 2) and theoretical (level 3) (van Hiele, 1986 cited in Teppo, 1991, p. 210). Many researchers elaborated on van Hiele levels and described the characteristics of every level (e.g., Burger & Shaugnessy, 1986; Pierre van Hiele, 1986; Crowley, 1987; Mason, 1998; Battista, 2007; Patsiomitou, 2012a).

They applied the van Hiele model to their investigations, determining the levels of thought and their characteristics and modifying the prototype version introduced by Van Hieles. Mason's (1998) first three levels of geometry understanding are the following (p.1):

Level 1 (<i>Visualization</i>):	Level 2 (<i>Analysis</i>):	Level 3 (<i>Abstraction</i>):
Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.	Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.	Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction, however, is not understood.

Battista (2007) "has elaborated the original van Hiele levels to carefully trace students' progress in moving from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians" (p.851). Battista's first three levels which are the most usual to high school students are described below:

Level 1 (*Visual-Holistic Reasoning*): "Students identify, describe, and reason about shapes and other geometric configurations according to their appearance as visual wholes. They may refer to visual prototypes, [...].Orientation on figures may strongly affect Level 1 students' shape identifications.[...]" (p.851).

Level 2 (*Analytic-Componential Reasoning*): "Students [acquire through instruction] a) an increasing ability and inclination to account for the spatial structure of shapes by analyzing their parts and how their parts are related and b) an increasing ability to understand and apply formal geometric concepts in analyzing relationships between parts of shapes". (pp.851-852). Battista (2007) identified three sublevels between levels 2 and 3, as follows:

2.1 <i>Visual-informal componential reasoning.</i>	2.2 <i>Informal and insufficient-formal componential reasoning.</i>	2.3 <i>Sufficient formal property-based reasoning.</i>
Students describe parts and properties of shapes informally and imprecisely.[...]	As students begin to acquire formal conceptualizations that can be used to "see" and describe spatial relationships between parts of shapes, they use a combination of informal and formal descriptions of shapes. [...]	Students explicitly and exclusively use formal geometric concepts and language to describe and conceptualize shapes in a way that attends to a sufficient set of properties to specify the shapes. [...]

Level 3 (*Relational –Inferential Property-Based Reasoning*): "Students explicitly interrelate and make inferences about geometric properties of shapes.[...]" (pp.852-853). Battista (2007) identified three sublevels between levels 3 and 4, as follows:

3.1 Empirical relations. Students use empirical evidence to conclude that if a shape has one property, it has another.[...]	3.2. Componential analysis. By analyzing how types of shapes can be built [...], students conclude that when one property occurs, another property must occur.[...]	3.3 Logical inference. Students make logical inferences about properties: they mentally operate on property statements, not images. [...] Such reasoning enables students to make the inferences needed for hierarchical classification. [...]
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In my opinion, van Hiele's description of level 2 corresponds to Battista's description of level 2.1; Battista's description of level 2.3 relates to Mason's level 3 and both relate to the development of students' ability to define geometric objects. Moreover, there is no stability in the process, but this depends on the geometry activities the student participates in, and on the teacher's instructions that lead to the evolution of each individual student's level.

Van Hiele described periods between levels. In these periods the students have characteristics of both levels. For example, during the first period (between the first and the second levels) the students' perceptual competence in relation to a geometrical object gradually transforms from a global perception of the object to the perception of an object with concrete characteristics and properties. The meanings of *symbol* and *signal* are very important in the van Hiele model. Skemp (1987) defines a symbol as "a sound, or something visible, mentally connected to an idea" (p. 47). Piaget (1952/1977) in his work "The origins of intelligence in children" (translated by Cook, M.) also, states:

"The "symbol" and the "sign" are the signifiers of abstract meanings, such as those which involve representation. A "symbol" is an image evoked mentally or a material object intentionally chosen to designate a class of actions or objects. So it is the mental image of a tree that symbolizes in the mind trees in general, a particular tree which the individual remembers, or a certain action pertaining to trees, etc. Hence the symbol presupposes representation [...]. Symbol and sign are only the two poles, individual and social, of the same elaboration of meanings" (p.191).

Dina van Hiele supports that (Fuys et al, 1984, p.215)

"The word 'symbol' should here be interpreted as meaning 'a mental substitute for a complex of undifferentiated relations that is subsequently elaborated in the pupil's mind.' The rhomb, for instance, is a symbol of the following characteristics: it has four equal sides, equal opposite angles, diagonals that bisect the angles and are perpendicular to each other".

What is important is the students' competence when it occurs to identifying a figure's properties (*symbol character*) and to gradually identifying a concrete figure from a set of properties (*signal character*): for example, when a student observes an equilateral triangle in his textbook, being able to identify the figure's congruent sides and angles. The equal sides and angles are the main characteristic of a triangle; this is a *symbol* for the equilateral triangle. Then s/he can identify additional properties (for example, "every angle of an equilateral triangle is equal to 60 degrees"). All these properties are interrelated

and can become a concept for the concrete mathematical object (i.e. the equilateral triangle mentioned above). Subsequently, the student can use a combination of properties to construct the equilateral triangle. In other words, the student now possesses the concept of the triangle: an abstract idea conceived in her/his mind. This is a *signal* for the concrete *figural concept*. Generally, in my opinion, a symbol is a mental image of a class of objects with concrete characteristics and properties. A sign is the social aspect of the symbol which was previously created in an individual's mind.

Students' conceptual understanding has to do with their understanding of abstract ideas (Rittle-Johnson and Schneider, 2014). Pieron (1957, cited in Fischbein, 1993, p. 139) defines concepts as "symbolic representations (almost always verbal) used in the process of abstract thinking [...]".

As a student's mind moves forward to van Hiele levels, s/he is able to interlink concepts to produce a meaning. As Fischbein (1993) points out:

"What characterizes a concept is the fact that it expresses an idea, a general, ideal representation of a class of objects, based on their common features. (p. 139) [...] When you draw a certain triangle ABC on a sheet of paper in order to check some of its' properties [...] you do not refer to the respective particular drawing but to a certain shape which may be the shape of an infinite class of objects (p. 141) [...] all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties" (p.142).

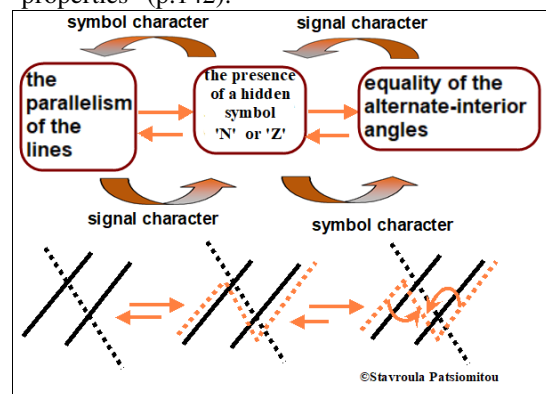


Fig. 2. An example of a diagrammatic illustration of students' interplay between symbol and signal character

Alternatively, the acquisition of students' signal character can be seen as their competency to reverse reasoning in their thinking (Patsiomitou, 2012 a, b). Dina van Hiele (Fuys et al. 1984) explains the meanings with the following example: "the parallelism of the lines implies (according to their signal character) the presence of a saw, and therefore (according to their symbolic character) equality of the alternate-interior angles" (p.218).

My students, for example, identify the letter "Z" or "N" (a hidden symbol) when they try to prove the equality of the alternate-interior angles (Fig. 2). If the students have the competency to reverse their reasoning, then they have also acquired the competency to form a proof, as they have the

competency to order logically their utterances (Patsiomitou, 2012a, b). According to Dina van Hiele (Fuys et al., 1984)

“On reaching this third level of thinking, which we call insight into the theory of geometry, we can start studying a deductive system of propositions [...]. Definitions and propositions now come within the pupils' intellectual horizon” (p.219).

Govender & De Villiers (e.g., 2002, 2004) clarified students' definitions as follows: (1) *Arbitrary definition*: a different, alternative but correct definition for the same concept. (2) *Sufficient definition*: It contains enough information [...] and only those elements of the set we want to define. (3) *Incomplete definitions*: It contains insufficient and incorrect properties (4) *Economical definitions*: It has only necessary and sufficient properties. For the use of my study I defined two more kinds of definitions students use (Patsiomitou, 2012a, 2013): *Arbitrary and economical definition* is a definition which is a synthesis of arbitrary (a different, alternative but correct definition for the same concept) and simultaneously it has only necessary and sufficient properties. In addition, a *dynamic perceptual definition* is the term by which the student informally 'defines' a geometrical object by using the tools of the software.

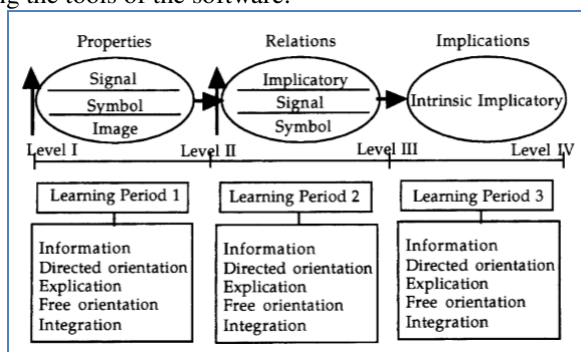


Fig. 3. Sang Sook Choi-Koh's (2001, p.302) van Hiele visual model of instruction

In his PH.D thesis Sang Sook Choi-Koh (2001) investigated the development of students' thinking, using The Geometer's Sketchpad software. He identified four learning stages in terms of symbol, signal and “implicatory” properties. He also used “active visualization”, meaning “the process of forming and interpreting geometric, dynamic representations within a computer environment” (2001, p, 302). Fig. 3 depicts Choi-Koh's van Hiele visual model of instruction. In my Ph.D (Patsiomitou, June 2007- December 2012), I also used the Geometer's Sketchpad to investigate how the van Hiele model can contribute as a base for the design of supportive materials in a classroom curriculum. In a short review I concluded that a mathematics education must be a process of *dynamic reinvention*, in a *Linking Visual Active Representations* (e.g., Patsiomitou, 2008b, c, 2010) frame. The synthesis of a *Dynamic Hypothetical Learning Path* (Patsiomitou, 2010, 2012a, b) as a process of design and redesign in the dynamic geometry software played an important role in the

construction of proofs and in the students' cognitive development.

IV. DYNAMIC GEOMETRY SOFTWARE: AN 'ALIVE' MICROWORLD

Dynamic geometry systems (DGS) are microworlds designed to facilitate the teaching and learning of Euclidean geometry, Algebra and Calculus. Microworlds have been described (Edwards, 1998, p. 74) “as ‘embodiments’ of mathematical or scientific ideas” that, in the words of Sinclair, & Jackiw (2007) “are extensible (so that the tools and objects of the environment can be built to create new ones), transparent (so that its inner workings are visible) and rich in representations.” (p.1)

A DGS software can play a fruitful and crucial role in the process of creating and evaluating conjectures which promote student creativity, and in so doing greatly contribute to developing mathematical reasoning. There are 2-dimensional DGS packages, such as the Geometer's Sketchpad (Jackiw, 1991/2001), Cabri II (Laborde, Baulac, & Bellemain, 1988), Geogebra (Hohenwarter, 2001), Cinderella (Richter-Gebert & Kortenkamp, 1999) etc. as well as 3-dimensional DGS packages, such as Cabri 3D (Laborde, 2004), Archimedes Geo3D (website [1]) etc. El-Demerdash (2010, pp. 23-26) reports many purposes and functions of a DGS software: (a) as a construction tool provides “an accurate constructor for creating geometric configurations and has the ability to automatically adjust and preserve the variant and invariant properties of constructed geometric configurations under dragging in a visual, efficient, and dynamic manner” (El-Demerdash, 2010, p. 23), (b) as a visualization tool (e.g., Straesser, 2002, 2003; Christou et al., 2005), (c) as a modeling tool (Oldknow, 2003; Patsiomitou, 2006b, 2008a, b, c, 2010, 2013, 2014, 2018), (d) as a tool for experimentation, exploration and discovery (e.g., Clements & Battista, 1992; Hollebrands, 2003; Kortenkamp, 2004; Patsiomitou, 2018), (e) as a tool for problem solving and problem posing (e.g., Sinclair 2002a, b; Christou et al., 2005; Patsiomitou, 2008a,b, 2014, 2018), (f) as a tool for teaching geometry with the utilization of transformations and the construction of proof (e.g., Hollebrands, 2003, 2007; Patsiomitou, 2008b, c, 2009, 2010; 2012a, b; Haj-Yahya, & Hershkowitz, 2013).

A first and very important effect on students' thinking stems from the Sketchpad software allowing the user to create sequential linking pages so that the whole Sketchpad file becomes an “*alive book*” (Patsiomitou, 2005, p. 63, in Greek; Patsiomitou, 2014). The “*alive digital representations*” (Patsiomitou, 2005, p. 67) function, which makes the whole figural diagram “*alive*”, giving the students *the potential to focus their attention on simultaneous modifications (and transformations) of objects on the screen* (Patsiomitou, 2005, p. 68), also yielded important results during my investigations. According to Sketchpad Help system

“Over time, you may want to add additional pages to a document. For example, you may want to organize a series of sketches that develop an argument; you may

want to present an activity that has several parts; or you may want to explore a conjecture in more depth than would be possible in a single sketch”.

A second important effect on students' thinking stems from the *dynamic transformations in a DGS environment*, a way of modifying an object on screen. We can change a figure's orientation, a figure's size or we can reconfigure it from its parts (Duval, 1995, 1999). Translations, rotations and reflections are the kind of transformations that preserve the size and shape of a figure. Any transformation (i.e. rotation, translation, reflection) of an object on screen produces a similar or congruent object image on screen. If we drag any point of the object the same transformation occurs to the image object that means that the image object (or reversely) follows the dragging results that refer to the object (e.g., Patsiomitou, 2009).

A third important effect on students' thinking occurs from dynamic constructions, that are the constructions created in a DGS environment. Daniel Scher (2002) in his study describes the characteristics of a traditional static construction in contradiction to a dynamic construction. The static constructions possess two characteristics as Scher (2002, p. 1) states: “they are static and particular”. In Scher's (2002) words “*the dynamic objects can be moved and reshaped interactively [...and] a single on screen image represents a whole class of geometric objects*” (p.2).

I supported the following from the empirical results of my investigations (e.g., Patsiomitou, 2011, 2012a): “The construction of a figure on screen in a DGS environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation (“construct a parallelogram” for example) into a mental image, which is to say an internal representation recalling a prototype image (e.g., HersHKovitz, 1990) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. This process requires the student to decode their actions using software primitives, functions etc. In order to accomplish a construction in the software the student must acquire the competence for *instrumental decoding* meaning the competence to transform his/her mental images to actions in the software. Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, based on Duval's (1995) semiotic analysis of students' apprehension of a geometric figure. Duval has distinguished three kinds of operations, one of which is the *place way*, meaning an operation which changes a figure's orientation. During the development of a construction, I think that the student has to develop three kinds of apprehension when selecting software objects which accord with the types of cognitive apprehension outlined by Duval (1995, pp.145-147) namely *perceptual*, *sequential*, *discursive*, and *operative apprehension*. In concrete terms, the competence of instrumental decoding in the software's constructions depends on: a) *the sequential apprehension* of the tools selection (i.e. s/he has to follow a predetermined order); b) *the verbal*

apprehension of the tools selection which means the student has to verbalize this process, and c) *a place way type of elements operation* on the figure (i.e. when s/he transforms the orientation of the elements to apply the command due to his/her perceptual apprehension). Then s/he has constructed *the operative apprehension* of the figure's elements for the construction, meaning the competence to operate the construction”.

A fourth important effect on students' thinking occurs from the construction of custom tools /scripts (e.g., Patsiomitou, 2005, 2006 a, b, 2007, 2008d, 2012a, b, 2014). As Straesser (2001) supports

“Apart from practical considerations (like exactness and ease), DGS-use can be structured according to conceptual units by means of macro-constructions. DGS-constructions are not bound to follow the small units of traditional drawing practice. Offering new tools that are unavailable in paper and pencil geometry, DGS-use widens the range of accessible geometrical constructions and solutions. If these tools become everyday instruments in the hands and minds of the user” (p.332).

During the construction of a custom tool a user determines the order the dynamic objects have to be created. This is in accordance with what Balachef & Kaput (1997) support that

“The order in which actions take place could become arbitrary in the eyes of users, which can have significant consequences. [...] This demonstrates the impact of the orientation of the plan which is in general forgotten in elementary geometry, but is recalled to the user as a result of the sequencing of actions (Payan 1992)”. (p.13)

I shall further discuss the meaning of custom tools in the next section.

The fifth [and most] important effect on student's thinking stems from the DGS software's dragging facilities. Sketchpad's dragging behavior transforms an object on screen moving that object on the screen.

For example, if we create a triangle on screen it can be dragged and transformed into an infinite number of figural-triangles that determine the concept of triangle in every change of orientation and shape. I consider there to be two main diacris in dragging utilizations with regard to students actions (Patsiomitou, 2011, 2012a, b): (a) *the theoretical dragging* in which the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties and (b) *the experimental dragging* in which the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure (or drawing).

Dragging an object in a DGS environment leads to the transformation of the object.

- The object (e.g., a rectangle constructed in a theoretical way) remains unaltered in terms of its structural characteristics, but the length of a side on

screen is transformed due to the manner of its construction (a 'visual way' transformation, in the words of Duval). The object's orientation also can be transformed in what Duval (1995) calls 'a place way' transformation.

- The object is messed up as a result of the non-theoretical way in which it has been constructed (its construction depends on the student-user's conceptual understanding).
- The object is restructured, remaining an invariant construction on screen, because it has been constructed in a theoretical manner (*a mereological way of shapes' reconfiguration*).
- The object is unaltered as it is dragged on screen from a point. It appears as a static object, but it remains intrinsically dynamic due to the dependence of the aforementioned point's parent objects. In my opinion, it is a *hybrid object*, which transforms the whole diagram to a *hybrid-dynamic representation*.

The transformation of an object on screen using dragging can be combined with other techniques to cause a combination of transformations on screen (e.g., Patsiomitou, 2008b, c, 2010, 2012a, b): (a) dragging and tracing objects (b) dragging and measuring objects (c) dragging and animating objects (d) dragging a transformed object or its image (by rotation, translation or reflection) or more complex such as (a) dragging, tracing and animation and (b) dragging, measuring and rotating etc. It is not within the scope of this paper to discuss the dragging facilities in any more detail, but like to Goldenberg & Cuoco (1996) I would argue that

"Dynamic Geometry *needs its own axiomatic foundation* to define the objects and postulates of its environment. (In particular, such a foundation would describe and, following Poncelet, properly mathematize the dragging transformation)" (cited in Jackiw, & Sinclair, 2009, p.415)

Generally speaking, a computer learning environment such the Geometer's Sketchpad scaffolds students' co-building of the meanings introduced in the teaching and learning activity. The design of activities in the learning environment (the software) as a part of the instruction thus has a crucial role to play in the comprehension of mathematical meanings. Jackiw, & Sinclair (2009) also argue that:

"[...] A Dynamic Geometry [object] is not an illustration, in other words— not an example of some more abstract, general, or encompassing idea—it is that idea and fully manifests its extent. At the same time, the dragged [object] implies a dragging intelligence. And this hidden actor, in whose hands the [object] *comes alive*, is the other focus of research attention" (p.414).

V. ARTIFACTS-[CUSTOM TOOLS] AND INSTRUMENTS-[CUSTOM TOOLS]

Straesser (2002, p.65) supports that "even if the DGS programs differ in their conceptual and ergonomic design, they share [...] the ability to group a sequence of construction

commands into a new command (macro-constructions". Kadunz (2002, p. 73) also in his study "Macros and Modules in Geometry" considers that among other characteristic features in a DGS environment are "macros" which condense a series of constructions steps into one software command. DGS share three main features, drag mode, locus of points, and the ability to define and use macros/scripts-custom tools, a kind of 'technical chunking'. Chunking "supports and facilitates cognitive processes involved in encoding, extracting, remembering, and understanding information" (Winn, 1993; Gobet et al., 2001 quoted in Sedig, & Sumner, 2006).

In my opinion, a custom tool is an encapsulation of a sequence of primitive objects and construction commands into a new tool, combining information of the construction in a consequential mode.

It combines in a concrete and sequential order the steps that have been used to accomplish the construction. For example, if we construct a square, we can save the concrete construction in a custom tool which can repeat the construction in the concrete way used by the creator of the custom tool, meaning that it processes the objects in the same sequence. The dragging of the custom tool constructed on screen follows the rules that refer to the primitives and commands incorporated into the custom tool (i.e. if we have measured angles or segments, or calculated a ratio, during our construction of the tool, then the concrete measures and calculations are repeated any time we implement the custom tool). If we drag the tool, the measures follow the increasing or decreasing of the length of the segments and angles (e.g., Patsiomitou, 2005, p.83).

According to Straesser (2001, 2002, 2003) macros [custom tools for the current study] "can help to structure a geometrical construction by condensing a complicated sequence of construction steps into one single command". This is in other words "a chunk of knowledge", as Simon (1980) points out, "A chunk is any perceptual configuration [...] familiar and recognizable" (Simon, 1980, p. 83) that helps the students to reverse their thoughts (e.g., Patsiomitou, 2012a). Researchers in cognitive psychology (e.g., Dörfler 1991; Dubinsky 1988; Frick, 1989) report that chunking information facilitates memory and retrieval. In a chunk, knowledge is condensed "into a unit available to the learner as a whole" (Kadunz, 2002, p.73). Weibell (2011) also states:

"One effective strategy that can be used to extend [or increase] the amount of information held in working memory is chunking (Miller, 1956). Chunking is a process of recoding multiple bits of information into a meaningful representation that contains the same amount of information, but takes up fewer slots in memory" (p.110).

By constructing a custom tool, we can help students to extend the capacity of their working memory, since the knowledge the student must retain is reduced. Nonetheless, the basic underlying notion is that a student is able to codify a construction and the concrete codification shape what the student can do when s/he will encounter a new situation related to the concrete that has been abstracted and codified

with the use of custom tool. A custom tool created in a DGS environment is a digital artefact. An 'artefact' custom tool, with which the interaction takes place during the mathematical activity, is transformed into an 'instrument' custom tool. An instrument (Rabardel, 1995) combines both an artefactual, material structure (external result) and a psychological schematic structure (internal result) directly linked to the use of the artifact (e.g., Artigue, 2000; Trouche, 2003, 2004).

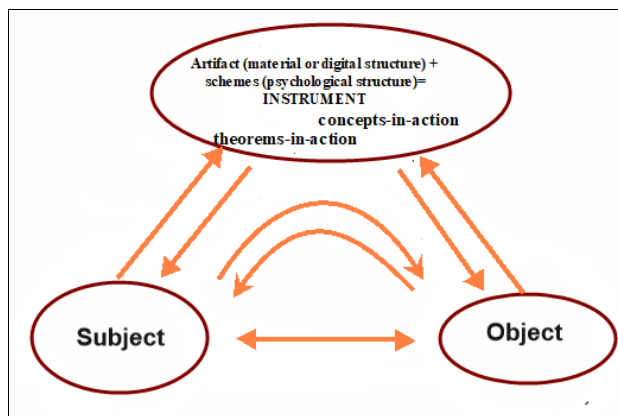


Fig. 4. The mediating instrument (Beguín & Rabardel, 2000, p. 179), an adaptation for the current study.

This is in accordance with what Beguín & Rabardel (2000) state with regard to structures an instrument is made:

- *psychological structures*, which organize the activity;
- *artifact structures*, which [...] are the signs and symbols in the code used to think of and express solutions, along with the paper, pencils, erasers, and so on, that serve to produce and modify the diagrams" (p.179).

During the research process, students discuss their ideas and make inferences in relation to the diagrams' dynamic transformations. According to Trouche (2003, 2004) a scheme of instrumented action constructed during the instrumental genesis process includes *theorems-in-action* ("propositions believed to be true" and *concepts-in-action* ('concepts implicitly believed to be relevant') (Vergnaud, 1998). It may help to analyze the students' work and to decompose the problem solving strategy. Students during the instructional period also develop different kinds of reasoning: *inductive*, *abductive*, *deductive*, *transformational* etc. Pedemonte's (2007) has defined the meanings of *abduction*, *induction* and *deduction* as follows:

"*Abduction* is an inference which allows the construction of a claim starting from an observed fact. *Induction* is an inference which allows the construction of a claim generalizing from some particular cases. *Generalization* plays an important role in inductive argumentation. *Deduction* is an inference allowing the construction of a claim starting from some data and a rule." (p. 29).

Pedemonte (2007, p.28) has presented Toulmin's (1958) basic structure of an argument constructing a figure with the three

basic elements mentioned above (Fig.5). For the representation of a theoretical diagram using tools and theoretical constructs I introduced a pseudo-Toulmin's model (Patsiomitou, 2011, 2012a, b) --based on Toulmin's model (1958) -- in which: (1) the data could be the dynamic diagram, or an object and (2) a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation).

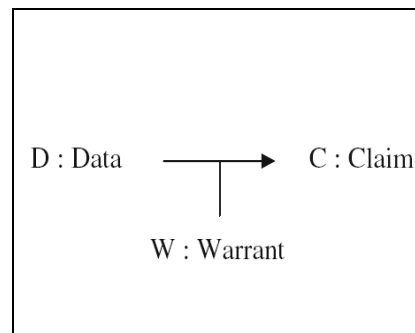


Fig. 5. Toulmin's (1958) basic structure of an argument (Pedemonte, 2007, p.28)

Students also many times use a tool (or a custom tool) in *an economical mode* or a *catachresis mode*. An *economical mode of the tool* is determined when a student tends to use a tool that previously has been used for a first task "to carry out a new task" Rabardel (1995, p.96). The idea of 'catachresis' in the words of Beguín & Rabardel (2000)

"is employed in the field of instrumentation to refer to the use of one tool in place of another, or to using tools to carry out tasks for which they were not designed" [...] catachresis [is] an indicator of the user's contribution to the development and use of an instrument. The existence of catachreses reveals that the subject creates means more suited to the ends he or she is striving to achieve, and constructs instruments to be incorporated into the activity in accordance with his or her goals" (p.180).

This means that a student can use a custom tool efficiently to discover the properties of a more complex figure. Also, s/he can use it in an economical mode or s/he can extend its mode by catachresis. This dynamic active functionality of the custom tool presupposes the student to act on the tool (external use of the construction) thus the tool is shaped by the user during the *instrumentalization process* while the artefact simultaneously acts upon the subject (internal use of the structure) and the tool affects and shapes the users' thought during the *instrumentation process* (e.g., Artigue, 2000; Trouche, 2003, 2004; Drijvers & Trouche, 2008; Patsiomitou, 2008b, c, d). Consequently, the student creates an adaptation of his older scheme (an accommodation of the user to the tool). The latter notion accords with the idea that a student accommodates a custom tool to his/her needs (Patsiomitou, 2008d).

VI. RESEARCH METHODOLOGY

I conceived and applied a re-conceptualized, hypothetical learning path for the teaching and learning of quadrilaterals in geometry (Patsiomitou, 2008b, 2010, 2012a, b), using the Geometer's Sketchpad environment as part of my Ph.D. thesis to raise the students van Hiele levels. As it is reported in previous studies (e.g., Patsiomitou & Emvalotis, 2009a, b, 2010, p. 34) I was responsible for the choice of activities (digital or material) included in my PhD learning trajectory, for session planning and students' assessment. In the current study, I shall report an excerpt from the research process I conducted after completing my Ph.D. My Ph.D research study lasted almost 5 months (January 2007-May 2007). I was curious to see whether I would have the same results if I compressed the outcome of the learning trajectory into a shorter period.

Firstly, I examined student's level of geometric thought using the test developed by Usiskin (1982) at the University of Chicago which is in accordance to the van Hiele model using only the first 20 twenty questions of the questionnaire. In the current study I shall present excerpts of the implementation of the custom tool in a pair of students M15, M16. M15 is a male student (van Hiele level: 2) and M16 is a female student (van Hiele level: 1).

Both students were volunteers for the experimental process, low achievers in mathematics, and good friends. It is important to mention that I devoted a short period (almost two weeks) to introduce the students M15, M16 to Sketchpad transformations, following the learning trajectory (Patsiomitou, 2010, p.1): Phase 1 – construction activities; Phase 2 – construction through symmetry activities; Phase 3 – the exploration of open-ended problems; and Phase 4 – building and transforming semi-pre-designed Linking Visual Active Representations (LVAR). The excerpt from the research for the current study relates to an episode midway through the second phase. What is described here lasted almost 30 minutes. I have reported the importance of the use of the custom tools in many previous studies (e.g., Patsiomitou, 2005, 2006a, b (in Greek), 2007, 2008a, d, 2012a, b, 2014). For the current study, I used a custom tool I had previously created to help students visualize the meaning of central symmetry in correlation with the meaning of a segment's midpoint. This was very crucial for the evolution of the construction of a parallelogram through its diagonals.

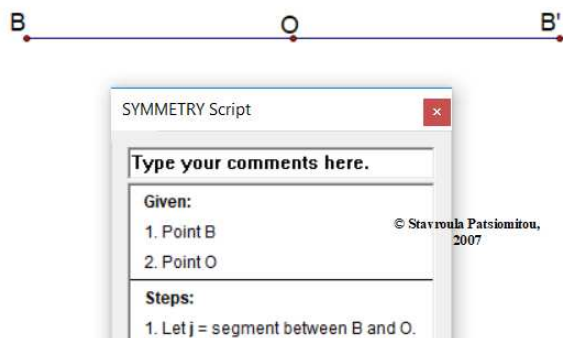


Fig. 6. The custom tool “symmetry”

The construction of the tool is very simple (Fig. 6), but it had a crucial effect on the development of the students' thinking (Patsiomitou, 2012a, b). The idea of creating the concrete custom tool occurred after creating a similar tool to construct the “golden ratio” (Patsiomitou, 2006b, 2008a, in Greek).

If my students implemented it on screen, they could view a segment with its midpoint. This tool helped them connect the meaning of symmetry by center with the meaning of a segment's midpoint. If they applied it twice on a point F, they visualized an “X” symbol which students view when constructing the diagonals of a parallelogram. Fig. 7 illustrates an implementation of the custom tool once on screen. In Fig. 8, I implemented the tool twice on point F. The modification of the orientation of the segment KK' (or the angle between the segments GG' and KK') determines the kind of parallelogram produced/generated (e.g., Patsiomitou, 2012a, b, p.72).

Firstly, a student-user assimilates the meaning incorporated in the use of the tool into his preexisting knowledge (for example s/he connects the meaning of the symmetry by center with the meaning of the segment's midpoint). S/he may then face an obstacle (an *instrumental obstacle*) (Patsiomitou, 2011, p. 362) with regard to the use of the tool. I distinguished a few types of *instrumental obstacles* due to student lack of competence in *instrumental decoding*. For example, the tool cannot be applied on a segment to find its midpoint. This occurs because I created the tool with concrete properties (Fig. 6) to incorporate the meaning of rotating a point by 180 degrees. This assumption generates a *cognitive conflict* in the student. The students used the tools efficiently or in an economical (*catachresis*) mode. In the current study, I shall present the impact of the custom tool on students' thinking, as well as the development of their abstract thinking, the recognition of instrumented action schemes through the emergence of theorems and concepts-in-action and the verbalizing of concepts during the process. The problem I posed the students was this: *Can you construct a rectangle using the properties of its diagonals?*

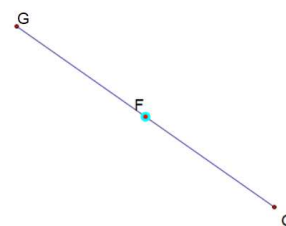


Fig. 7. Implementation of the custom tool “symmetry”

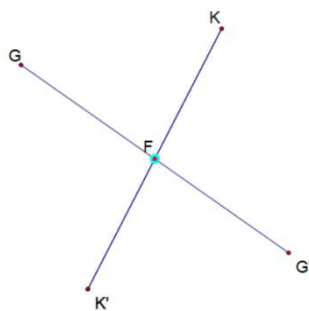


Fig. 8. The “X” utilization scheme of the custom tool

The methodology used, consisted of a case study of a pair of students, action research (Bogdan & Biklen, 1998) and “theory-building” (Eisenhardt, 2002) from the case study. I used qualitative analysis to investigate potential improvements using the Sketchpad dynamic geometry software and to assess the impact of the DGS interventions in the curriculum on students’ geometrical thinking. I analyzed the data using grounded theory’s constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1990). I videotaped the students’ discussion and analysed them simultaneously with my fieldnotes during the research process. I recognised in van Hiele’s theory a framework appropriate for the description of my research analysis. Furthermore, in a didactic experiment conducted in the domain of learning processes the researcher is responsible for the pedagogical nature of the activities s/he uses. Generally speaking, s/he must also have in mind that “undesirable factors cannot be eliminated” (Dina van Hiele-Geldof in Fuys et al., 1984, p. 13). Dina van Hiele-Geldof also pointed out the involvement of psychological phenomena during the instruction process. For this “the subjective attitude of the observer is a factor that is almost impossible to eliminate” (p. 13)

VII. THE EXPERIMENTAL PROCESS

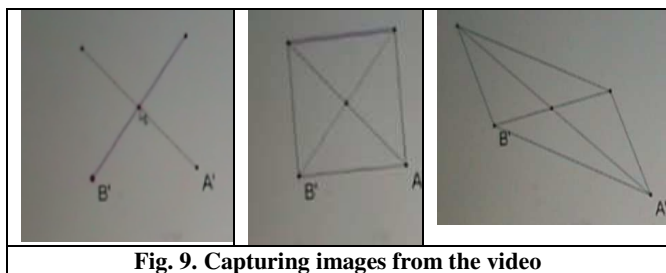


Fig. 9. Capturing images from the video

[1] M16: A parallelogram .What kind of parallelogram?

M16 constructed two intersected segments using the custom tool.

M16: Then we shall join these. [sides] ...but,... it is not a parallelogram!

[2] Researcher: What are the prerequisites for a quadrilateral to be a parallelogram?

[3] M15: The opposite sides must be congruent; the diagonals must be dichotomized.....

[4] Researcher: What can you view in the current situation? Do these segments dichotomize each other?

[5] M15: Yes!

[6] M16: They are congruent! (She moves the figure using dragging.)

[7] M16: They are congruent! It is a parallelogram! (She meant the half segments of a parallelogram’s diagonal).

[8] Researcher: ok...it is a parallelogram ...Can you construct a rectangle?

[9] M15: Well,...an angle bisector ... (pointing to a diagonal)

[10] Researcher: Diagonal, you mean!

[11] M15: Yes ...they must be dichotomized and.... they must be congruent.

[12] Researcher: Correct both! Can you construct it?

Fieldnote 1: The students constructed a parallelogram using the scaffolding effect provided by the tool. This point in the research is quite similar to other situations I faced in my previous studies with different pairs of students. The students faced a *cognitive conflict* because they could not use the terminology accurately. Most of them confused the meaning of angle bisector (‘dichotomos’ in Greek) with the meaning of ‘diagonal’. This confusion did not help them when they had to solve a problem, because, while the diagonals do also dichotomize the angles of the vertexes in a few quadrilaterals (i.e. rhombus, square), this is not the case in other quadrilaterals (i.e. parallelogram, rectangle, and trapezium). This confusion grew during the construction of a figure - parallelogram. Moreover, the students have to differentiate the angle bisector of an angle from an angle bisector of a triangle (to an angle bisector of a parallelogram). M15 can recognize and name properties of the parallelogram, but he still does not see relationships between these properties (Mason, 1998, level 2). In the concrete case M15 defines the object with a *dynamic and economical definition*. This is a sign that the student is moving to the van Hiele level 3. M16 makes decision based on perception. She recalls the structure of a parallelogram’s diagonals. M16 recognizes a property of the parallelogram from the ‘alive’ [active] representation on screen. M16’s pretest level was 1; this is clear from her answers, as she makes decisions based on perception.

[13] M15 constructed a segment with the custom tool trying to visualize as a diagonal of a rectangle ...He stopped and looked at it on screen.

[14] M15: I shall construct it as we constructed the parallelogram.

[15] Researcher: What should the rectangle’s diagonals be?

[16] M15: Congruent ...I shall construct a segment with the tool...

[17] Researcher: So, how can you construct a diagonal equal to this one?

[18] M15: I shall rotate it.

[19] M16: Construct a point ...not on the segment! Choose it and rotate the segment...

[20] M16: We should have 90 degrees...

[21] M15: Yes! I got it!

[22] M16: Let’s draw a straight line.

[23] M15: *We can construct a straight line ...we shall construct its midpoint (it looks like he wants to apply the custom tool to find the midpoint of the segment).*

[24] M15 selects the segment and its endpoints and tries to construct the midpoint from the menu.

[25] M16: *Why are you doing this? The tool (meaning the custom tool) can construct the midpoint.*

[26] M15: *Eureka! I shall construct parallels from these points (he selected the endpoints of the segment).*

[27] M15: *I shall join these two points.*

[28] M15: *Then I shall construct the symmetrical triangle by 180 degrees (Fig. 12)*

[29] M15 selected the midpoint and constructed a rotational symmetry of the triangle.

[30] M15: *Ok! It is readyyyy!*

[31] M16: *Is it a rectangle? Drag this point.*

[32] M16: *Choose a vertex to drag!*

[33] M16: *It is a very nice parallelogram! (laughing) ...but you went to Trikala and back when you were constructing it (a Greek expression for when a person follows a less than easy and obvious path when carrying out a task).*

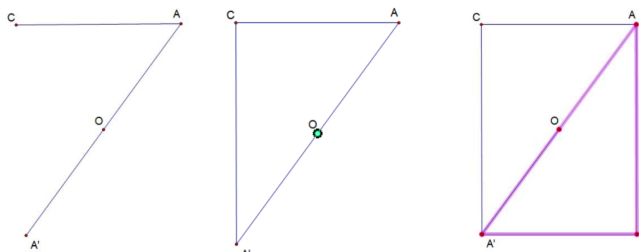


Fig. 10, 11, 12. Constructional steps of a M15' parallelogram

Fieldnote 2: M15 started with the construction of a segment using the custom tool. Then he constructed a segment AC and joined the point C with the point A'. He tried CA' to seem vertical to CA. Then he rotated the triangle CAA' by 180 degrees. His construction of the parallelogram (Fig. 10, 11, 12) is complex. M15 knows the properties of the figure "rectangle", but cannot implement them to construct it. He cannot "instrumentally decode his words to a figure on screen" (Patsiomitou, 2011, 2012a, b). He had to bring a perpendicular line down to the segment CA. He was familiar with the procedure for constructing a perpendicular line to a point on a segment, but he did not use it. On the other hand, he constructs a "parallelogram" figure using a reconfiguration of a triangle. The rotation of the triangle by 180 degrees could be the definition of a parallelogram when we use rotational transformation. M15 uses a combination of informal and formal descriptions of shapes (Level 2.2. according to Battista's classification). He knows that the rotated segments are congruent [point of the dialogue 18]. M15 is beginning to acquire formal conceptualizations that can be used to "see" and describe spatial relationships between parts of shapes. M16 is trying to use the tool in a catachresis mode, as she has extended the properties of the tool in her mind. The [alive] tool has affected her thoughts, as she has constructed an instrumented action scheme [point of the dialogue 25].

[34] M16: *Give me the mouse (laughing)...we shall construct a line ...we shall rotate this point by 180 degrees.*

[35] M15: *This is a parallelogram again.*

[36] M16: *But ...its diagonals are congruent!*

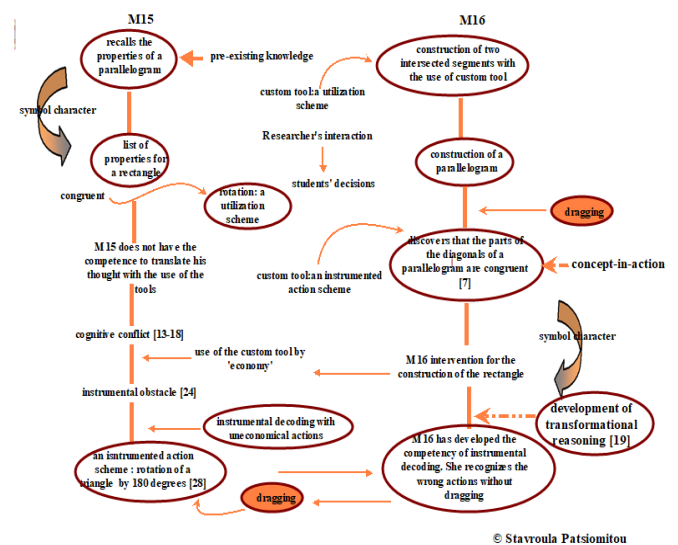
[37] M15: *Why? ...you can measure them ...drag them now...*

[38] Researcher: *How can you construct a segment congruent to this one?*

[39] Both: *we can rotate it ...or reflect it ...They will be symmetrical.*

[40] M16: *Eureka! We can do it!*

[41] M16: *We shall select this point (means the midpoint) ...we shall select this endpoint and we shall rotate it by 90 degrees.*



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Fig. 13. An illustration of the use of the tools using a pseudo-Toulmin's diagram

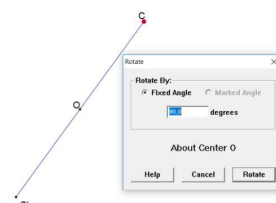


Fig. 14. Use of the tool and rotating point C through 90 degrees

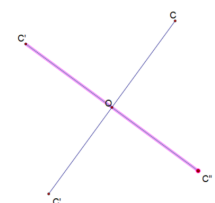


Fig. 15. Using the 'X' scheme formed by a parallelogram's diagonals to construct a square.

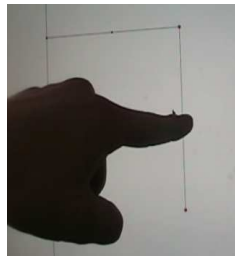
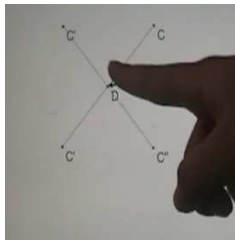


Fig. 16, 17. Students' gestures during the research process (capturing images from the video)

Fieldnote 3: M16 rotates point A through 90 degrees. She then uses the custom tool, applying it to points A' and O. She insisted that the diagonals are congruent (point [45]), but as M15 was not convinced by the dragging facility, she measured the segments and dragged them again using a combination of transformations. She ended up constructing a square when trying to construct a rectangle, as during the *instrumental decoding* she constructed a point A' in a concrete position ($A'O = OA$ and $A'O$ is perpendicular to AO). The most important conceptual event occurs ([49]) when she expresses a logical hierarchy regarding the inclusion of the rectangle and the square.

[42] M16: *Now I shall do it with the ease way ...*

[43] M16 selects the custom tool and applies it to the point and to the midpoint.

[44] M15: *Againnnnn, it is a parallelogram!*

[45] M16: *Why? Its diagonals are congruent!*

[46] They select them and measure them.

[47] M16: *It is a rectangle!*

[48] Researcher: *What is it? Drag all the vertexes!*

[49] M16: *...may be it is a square ... but the square is also a rectangle...so it is ok! I constructed it!*

[50] M15: ***The square is a rectangle???*** What does she say?

[51] I did not explain or mention why the square is also a rectangle, but posed one more question.

[52] Researcher: *Can you construct a rectangle? Not a square.*

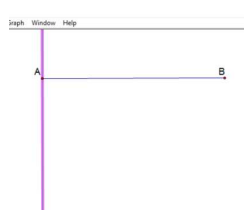


Fig. 18. Step 1

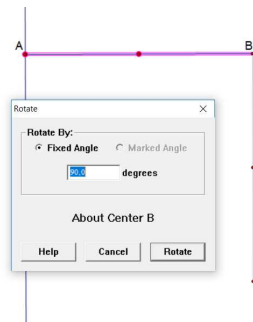


Fig. 19. Step 2

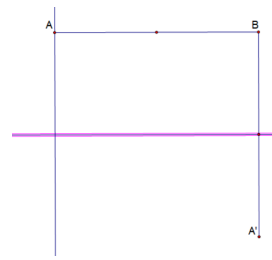


Fig. 20. Step 3

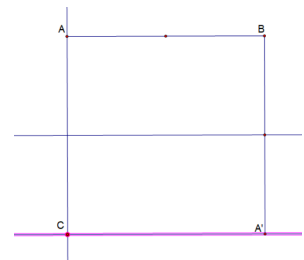


Fig. 21. Step 4

[53] M15: *I can do it!*

[54] M15 constructs a segment AB.

[55] M15: *Now we shall construct a perpendicular to this point (point A)*

[56] M15 then constructs the midpoint of the segment. *...I shall rotate only the half segment by 90 degrees... Oh, eureka!!*

M15 rotates the whole segment AB about center B by 90 degrees.

[57] M16: *You constructed a square again!*

[58] M15: *No!*

[59] M16: *Yesss! This segment is congruent to this segment!*

[60] M15: *Ok! We shall construct a parallel line from this point (A'). This will be a rectangle...*

[61] M16: *This is a square as all its sides are congruent and perpendicular (she means to one another)*

[62] M16: *We shall construct a rectangle ...we shall delete all these lines.*

[63] M16: *We shall construct a point here (point C)... We shall select this point (the midpoint), double click on it, and we shall rotate point C through 180 degrees*

(She stopped for a moment) *....then we shall join these points ...oh! No! ...it is not a rectangle!*

[64] M16 selects the custom tool and implements it efficiently at the points C' and B.

[65] M16 selects the custom tool again and implements it at points C'' and M...

[66] Researcher: *You constructed a rectangle!*

[67] M16: *Yeahhh!*

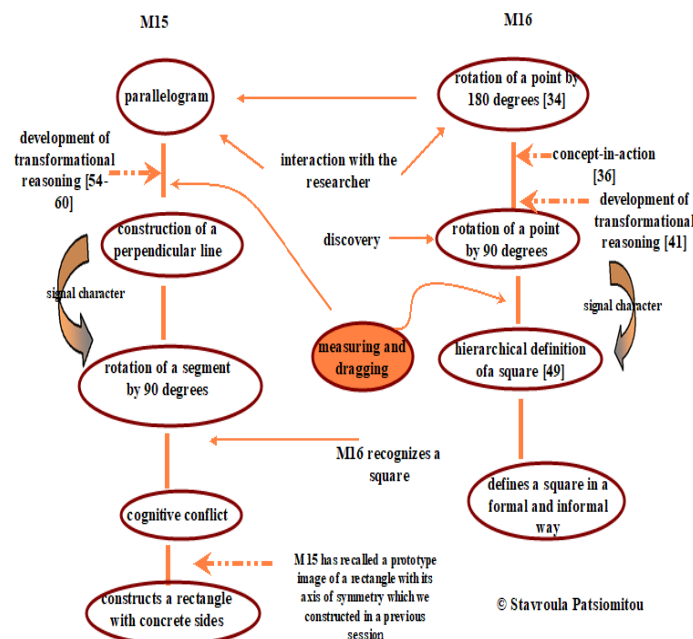


Fig. 22. An illustration of the use of the tools using a pseudo-Toulmin's diagram

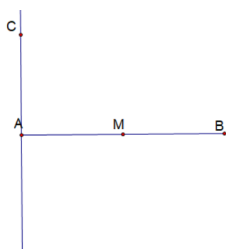


Fig. 23. Construction of an arbitrary point C

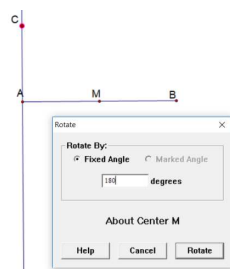


Fig. 24. Transforming point C through 180 degrees about center M

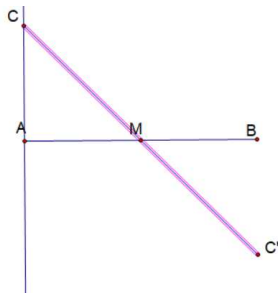


Fig. 25. Construction of the point C'

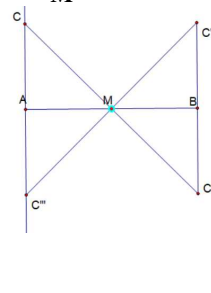


Fig. 26. Using custom tool to points C'' and M

Fieldnote 4: M15 tried to construct a rectangle. He has recalled a prototype image of a rectangle with its axis of symmetry which we constructed in a previous session. He ultimately constructed a rectangle whose side is half the length of the side of the square $ABA'C$. He is in transition to Level 3, but still lacks the competency to instrumentally decode a figure. M16 did not delete all the lines. She had something in

mind while M15 constructs his specialized kind of rectangle. She was not sure about the next step, but no one could take the mouse from her hand. She implied that BC' is a perpendicular line, as she constructed point C' by rotating point C , and she implied that CC'' is perpendicular to CA . She did not prove the sequential steps using deductive reasoning, but the construction steps she follows is an indicative of the development of abstract thinking. M16 developed what Simon (1996) calls *transformational reasoning*. What is transformational reasoning? In the words of Simon (1996):

"Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated" (p. 201)

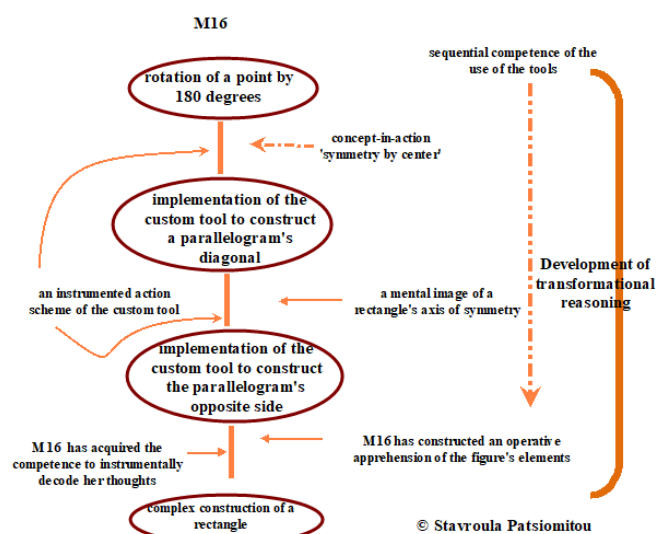


Fig. 27. M16's construction in a diagrammatic representation

Fieldnote 5: M16's conception of the meaning of the rectangle and the rectangle's *instrumental decoding* was **the most incredible I have ever seen** a student display when using the concrete tool. While a concept is an idea shared and accepted by the mathematical community, a student's conception refers to a student's explanation of a concrete concept. In other words, it relates to with the way the student shapes the idea in his/her mind. M16 made many transformations in her mind in order to construct the rectangle. She constructed an arbitrary point C and she rotated it by 180 degrees. She implied the congruence of the triangles CAM , MBC' , and subsequently the congruency of the segments $C'B$, CA . In order to construct a segment equal to the segment $C'B$, she used the tool. In other words, she constructed a conceptual object in her mind in which she encapsulated the properties of the tool. The implementation of the tool once again to construct the diagonal $C''C'''$ is a strong indication that she

was absolutely sure the diagonals of the rectangle would be congruent. She used the tool appropriately and efficiently (not with economy or catachrese). Moreover, she displays sequential place-way and verbal competency when using the tool. All these are strong indications that she has developed abstract thinking. Regarding my interaction with the students, I believe it was the necessary for the students to move on during the process. As Burkhardt (1988) notes,

“[...] the teachers must perceive the implications of the students' different approaches, whether they may be fruitful and, if not, what might make them so. pedagogically [also] the teacher must decide when to intervene, and what suggestions will help the students while leaving the solution essentially in their hands, and carry this through for each student, or group of students, in the class” (Burkhardt, 1988, p. 18).

VIII. DISCUSSION AND SUMPERASMATA

When analyzing the students' dialogues, I used the meanings I introduced in my description of the theoretical underpinning. Duval's (1999) theory views students' perceptual apprehension as complementary to Vergnaud's (1998) theory of operational invariants in the context of a process of instrumental genesis. Fou-Lai Lin & Kai-Lin Yang (2002) also support that

“While Duval's cognitive architecture, an organization of several systems, put emphasis on multifunctional registers, Vergnaud's cognitive theory of practice put emphasis on the mechanism of conceptual field. Their perspectives on cognition seemed complementary for analyzing how subjects developed definitions and propositions of geometrical figures. Duval supported us a framework of perceptual categories to describe conversion and coordination between different registers, and Vergnaud supported us a framework of mental organization to explain cognitive mechanisms” (p. 20).

Concretely, I think Fig. 28 represents the connections among the meanings included in the theoretical underpinning I described.

With regard to the concept of rectangle

The rectangle is a fundamental meaning in parallelograms. Students are able to recognize the prototype image of the rectangle from the first classes of primary school. The obstacles regarding the prototype image of the rectangle have broadly been discussed (e.g., Hasegawa, 1997; De Villiers, 1994; Laborde, 1994; Fischbein, 1993; Parzysz, 1991; Sfard, 1991; HersHKovitz, 1990 in Monaghan, 2000).

When the students were attempting to construct the rectangle, they tried to interpret their mental prototype image of the rectangle. At the beginning of the process they did not analyze the conceptual properties of the figures (e.g., perpendicular lines and congruent diagonals). Rather, what they saw was a holistic “material entity”—in other words, a “drawing” of a rectangle, exactly as Laborde (1993) defines it, rather than the

‘theoretical object’ (a ‘figure’ in the words of Laborde). Laborde (1993) comments: “At a low level the figure is viewed as an entity but not analyzed into parts or elements: all parts of the drawing must move together” (p. 66).

Thus the concept of ‘rectangle’ which the students constructed is a combination of a perceptual image plus conceptual properties from an appropriate scheme which they constructed in interaction with the software's ‘alive’ [active] tools.

Vygotsky (1978), supports that cognitive conflict is important to successful concept formation. This importance can lead teachers to anticipate the ways that will “reveal valuable insights into their conceptual understanding and will thereby assist teachers in their efforts to scaffold their students' learning” (Monaghan, 2000, p. 179).

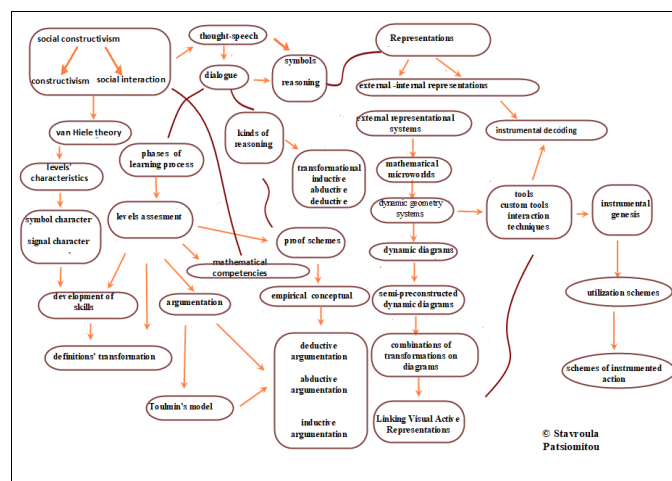


Fig. 28. Connections among the meanings included in the theoretical underpinning (Patsiomitou, 2012a)

With regard to dynamic transformations

Whiteley (1999) supports that “the ‘modern’ definition of geometry, due to Felix Klein in 1870, is [...] that geometry is a space with a group of transformations into itself (p.12). [For this] reasoning with transformations should be a central theme of our learning of geometry” (p. 15).

When we rotate a segment, a new image –object is created, congruent to the original segment. During the rotation of a segment by 90 degrees in the software a student constructs a utilization scheme of the tool which leads the students to conceptually grasp a) the meaning of perpendicularity between the segment and the image-object, in other words the existence of a right angle; b) the meaning of congruency between segments. This transformation has a significant impact: an instrument which includes a mental image of the rotation, since any modification of the initial figure (input) results in the modification of the final figure (output). If a student rotates points (not segments), s/he mentally constructs the lines that connect these points. This is a sign that the student has developed his/her abstract thinking. For example, a strong indication that M16 has mentally visualized her construction [see the dialogue from 62 to 67] is that **she rotates points, not segments or triangles**, as M15 did. I am going to explain it with an example. Suppose that a student has constructed a

segment AC and rotated it about point A through 90 degrees. The student visualizes the result, which is the segment AC'. M16 only rotated point D, as she had mentally visualized that "if we join point B with the point D' the angle will be equal to 90 degrees". This is a strong indication of her abstract thought. She has constructed a **figural concept of the rectangle in the words of Fischbein (1993)**.

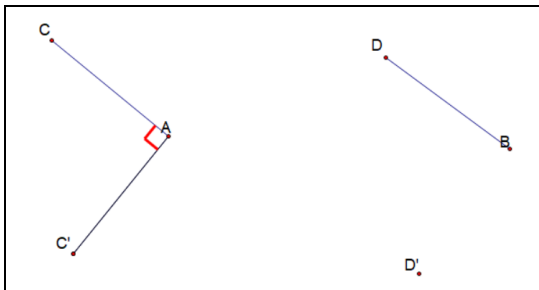


Fig. 29. Rotation of a segment vs rotation of a point

Following Kress' (1994) opinion that "*the child's language may be regarded as a window to the child's conceptual world*" (p. 132), I would complementarily add that a students' competency to transform a mental object to a figure on screen is a window to student's conceptual understanding.

With regards to students' symbol and signal character

During the study, students applied the tools and constructed what Rabardel (1995) calls *utilization schemes* of the tool/artefact. This process led to the development of *schemes of instrumented action*. The students assimilated the figures properties and are in the stage to perceive and accommodate the interrelationships between the properties of the figures. The next step is to use deductive processes and understand class inclusions (e.g. M15 at line [50]). This result will occur when the students have transformed the figures' symbol character into figures' signal character—a transformation that corresponds to the third level of geometric reasoning. Both students and particularly M16 developed efficient strategies to use the tools. M15's actions are the reverse of the actions he used to construct the axis of symmetry of a rectangle. He then constructed the symbol character of the rectangle. He did not make a rectangle with arbitrary sides, but rather a concrete rectangle. This is an indication that he is in transition to level 3, as he had not constructed the signal character of the rectangle. M16 has developed the competency to reverse her thoughts through the competency to make complex use of the tools to instrumentally decode the properties of the figures. The symbol character of the figures anticipated in her thought. For this, she has constructed the interrelationship between the meanings of the "parallel line in the middle of the distance of two parallel lines" with the meaning of axis symmetry and the meaning of the congruency of the diagonals of a rectangle. She does not express her thoughts in words, but she has been sufficiently developed the rectangle's and square's signal character. As Pierre van Hiele writes (1986, p. 168) a figure gets the "symbol character" when it becomes the representative of its properties as a unity. In my opinion, when

the student is able to reverse his/her thoughts and to anticipate the symbol of the figure, then the figure has received its signal character. The student can now list the similarities and differences between figures. S/he can also explain why a characteristic is not included in figures' characteristics.

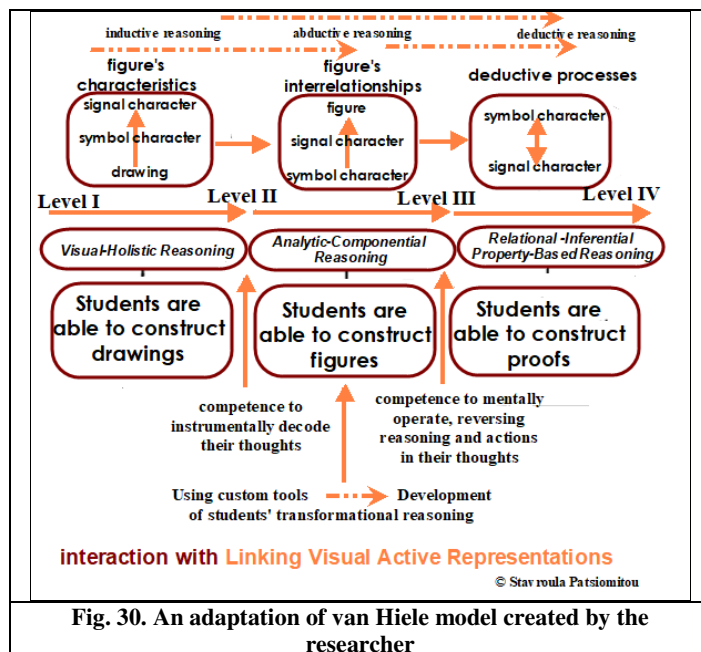


Fig. 30. An adaptation of van Hiele model created by the researcher

With regard to the students' van Hiele level

Fig. 30 illustrates an adaptation of the van Hiele model in relation to Choi-Koh's (2001) and Battista's (2007) levels of thinking. To clarify, when a student interacts with figural materials (for example a digital figure in a DGS environment), s/he interacts with the figure's characteristics: the equality of a square's sides and angles, the perpendicularity of a kite's diagonals, etc. Now s/he has in his/her mind which of these characteristics determine the concrete figure. During the second period of instruction s/he acquires a gradual competency to construct figures and during the third period of instruction the students are able to gradually construct proofs. In the current study, the participating students constructed: (a) the utilization scheme of the symmetry by center in correspondence with the midpoint of a segment; (b) the "X" utilization scheme of the custom tool, which was very important for the construction of a broader scheme, namely the instrumented action scheme of "the diagonals of a parallelogram". In Gawlick's opinion (2005) in a dynamic approach "the students can explore the topic in a directed orientation phase and then build the new concepts for themselves, drawing on their previous knowledge"[...] "so students get accustomed to the tools as well as to a "discoverer's" habit of mind"(p.370).

With regard to custom tools

In the present study, both students constructed schemes of instrumented action as a result of the efficiently use of the custom tool or its use in an economical mode. They extended

its use in a catachresis mode to construct the midpoint of a segment. This served for the construction of meaningful mental schemes to solve the problem. Consequently, the 'instrumented action' scheme, which is based on the construction and use of the custom tool, led students to construct mental objects. In the sequence of mental activities the students followed, mathematical knowledge and knowledge of the tool were combined. They constructed a first order instrumented action scheme and shaped the meaning "symmetry of point by 180 degrees", then a second order IS and shaped the meaning "the diagonals of parallelogram are dichotomized". The use of a combination of transformations using dragging and measuring or the rotating and/or implementing custom tools) helped them to shape the figural concept first of "the parallelogram", then of "the square", and finally of "the rectangle". The implementation of the custom tool helped students to shape a schematic entity in terms of their perceptions, and then led them through various stages to more abstract levels of cognitive perception. This also agrees with Edelman's viewpoint (1989/1992): "in forming concepts,...the brain areas responsible for concept formation contain structures that categorize, discriminate, and recombine the various brain activities occurring in different kinds of global mappings" (quoted in Davis & Tall, 2002). This means that custom tools can serve as structural units of knowledge, as conceptual objects and hence as 'schemes', too, including the structure and function of the encapsulated objects. The following statement is something I strongly propose as complementary to something I stated in a previous study (Patsiomitou, 2008d): *Custom tools are 'alive' encapsulated objects created in a DGS environment that operate as a referent point for organizing, retrieving and reversing information, and thus facilitating the anticipation and manipulation of the instrumented action schemes during an instrumental genesis process. A custom tool can become a medium for students' cognitive development and to develop their abstract thought.*

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