International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

Vol. 39 No. 1 June 2023, pp. 486-489

The Transformation Matrices of vec A to vec A^T for Diagonal Matrix

Rusdi Ahmad[a)] Yanita Yanita[b)], Lyra Yulianti[c)]

Department of Mathematics and Data Science, Faculty of Mathematics and Natural Sciences, Andalas University, Kampus Unand Limau Manis, Padang 25163, Indonesia

[a)]Corresponding author: rusdiahmad979@gmail.com
[b)]yanita@sci.unand.ac.id
[c)]lyra@sci.unand.ac.id

Abstract – This study aims to discusses the transformation matrices of *vec A* to *vec* A^T for diagonal matrix. The matrix used is a 4 x 4 diagonal matrix assuming that two pairs of elements are located on the main diagonal with the same value. To get the transformation matrices, this uses the commutation matrix equation is $K_{mn}\, vec(A) = vec(A^T)$. Therefore, several K_{mn} commutation matrices are obtained on a 4 x 4 diagonal matrix.

Keyword – ***Commutation Matrix, Diagonal Matrix, Vec Matrix.***

I. Introduction

The vec-operator transforms a matrix into a vector by stacking its columns one underneath the other. Let A be an m×n matrix. Then, two vectors vec A and vec A^T are the mn×1 vector, which contain the same mn components, but in a different order. Hence, there exist the transformation matrix that transform vec(A) into vec(A^T). This $mn \times mn$ matrix is called the commutation matrix, denoted by K_mn [1]. A commutation matrix is a kind of permutation matrix of order mn expressed as a block matrix where each block is of the same size and has a unique 1 in it [2].

This paper discusses the transformation matrices of vec A to vec A^Tfor diagonal matrix. The matrix used is a 4 x 4 diagonal matrix assuming that two pairs of elements are located on the main diagonal with the same value. Since the form of the matrices in this group can be classified based on the location of the elements on the diagonal of the matrix, several commutation matrices are obtained.

II. Research Methods

The research methods are based on the study of literature, which is related to the transformation matrices of vec A to vec A^Tfor diagonal matrix. The matrix used is a 4 x 4 diagonal matrix assuming that two pairs of elements are located on the main diagonal with the same value. In this Section, we present some of the definitions and properties (theorems) used to obtain the result.

Definition 2.1 [3]

A square matrix in which all the elements off the main diagonal are zero is called a diagonal matrix

Definition 2.2 [1]

The vec-operator transforms a matrix into a vector by stacking its columns one underneath the other. Let A be an m×n matrix and a_iits i-th column. Then vec A is the mn×1 vector.

Definition 2.3 [4]

A permutation of a set S is a function from S to S that is both one-to-one and onto.

If σ is a permutation, we have the identity matrix as follows:

Definition 2.4 [5]

Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma) = [\delta_{i,\sigma(j)}]$ where

$$\delta_{i,\sigma(j)} = \begin{cases} 1 & if\ i = \sigma(j) \\ 0 & if\ i \neq \sigma(j) \end{cases}, \delta_{i,\sigma(j)} = entry_{i,j}(P(\sigma)).$$

Theorem 2.5 [6]

Let π and σ be a permutation in S_n, then $P(\pi)P(\sigma) = P(\pi\sigma)$.

Definition 2.6 [2]

A permutation matrix $P = (p_{ij}) \in R^{n\times n}$ is called a commutation matrix if it satisfies the following conditions:

a) $P = (p_{ij})$ is an $p \times q$ block matrix with each block $P_{ij} \in R^{q\times p}$.
b) For each $i \in [p], j \in [q], P_{ij} \in \left(k_{st}^{(i,j)}\right)$ is a (0,1) matrix with a unique 1 which lies at the position (j, i).

The *vec-operator* transforms a matrix into a vector by stacking its columns one underneath the other. Let A be an $m\times n$ matrix. Then, two vectors $vec\ A$ and $vec\ A^T$ are the $mn\times 1$ vector, which contain the same mn components, but in a different order. Hence, there exist the transformation matrix that transform $vec(A)$ into $vec(A^T)$. This $mn \times mn$ matrix is called the commutation matrix, denoted by K_{mn} and defined implicitly by the operation $K_{mn}\ vec(A) = vec\ (A^T)$. The order of the indices matters: K_{mn} and K_{nm} are two different matrices when $m \neq n$, except when either $m = 1\ or\ n = 1$. If $m = n$ we write K_n instead of K_{nn} [1].

III. Results And Discussion

By using the properties of the commutation matrix, it is result that there are four commutation matrices for the 4 x 4 diagonal matrix. The matrix used is a 4 x 4 diagonal matrix assuming that two pairs of elements are located on the main diagonal with the same value. The following is the theorem generated in this research.

Theorem 3.1:

Let A be a 4 x 4 diagonal matrix

1) For any diagonal matrix A, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ is I_{16}
2) If $a_{ii} = a_{jj}$ elements, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ is $P(5i - 4 \quad 5j - 4)$
3) If $a_{kk} = a_{ll}$ elements, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ is $P(5k - 4 \quad 5l - 4)$
4) If $a_{ii} = a_{jj}$ and $a_{kk} = a_{ll}$ elements, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ is $P(5i - 4 \quad 5j - 4)(5k - 4 \quad 5l - 4)$

Proof:

Let A be a 4 x 4 diagonal matrix, that is

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

then

$$vec\ (A) = [a_{11} \quad 0 \quad 0 \quad 0 \quad 0 \quad a_{22} \quad 0 \quad 0 \quad 0 \quad 0 \quad a_{33} \quad 0 \quad 0 \quad 0 \quad 0 \quad a_{44}]^T = vec\ (A^T)$$

Note that, the a_{11} element in matrix A is the first element in $vec\ (A)$ and $vec\ (A^T)$, the a_{22} element in matrix A is the sixth element in $vec\ (A)$ and $vec\ (A^T)$, the a_{33} element in matrix A is the eleventh elementin $vec\ (A)$ and $vec\ (A^T)$, and the a_{44} element in matrix A is the sixteenth element in $vec\ (A)$ and $vec\ (A^T)$. So, the $[a_{ii}]$ elements places in the $1^{st}, 6^{th}, 11^{th}$ and 16^{th} element in $vec\ (A)$ and $vec\ (A^T)$. The numbers of 1, 6, 11, and 16 can be expressed by $(5i-4)$, for $i = 1,2,3,4$. Furthermore, if the elements $a_{ii} = a_{jj}$ and $a_{kk} = a_{ll}$, then the K_{mn} commutation matrix will be determined that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$

a) In this case, for any diagonal matrix A, the a_{ii} element places the $(5i-4)^{th}$ element in $vec\ (A)$ into the $(5i-4)^{th}$ element in $vec\ (A^T)$. The a_{jj} element places the $(5j-4)^{th}$ element in $vec\ (A)$ into the $(5j-4)^{th}$ element in $vec\ (A^T)$. The a_{kk} element places the $(5k-4)^{th}$ element in $vec\ (A)$ into the $(5k-4)^{th}$ element in $vec\ (A^T)$. The a_{ll} element places the $(5l-4)^{th}$ element in $vec\ (A)$ into the $(5l-4)^{th}$ element in $vec\ (A^T)$. So we have the commutation matrix is $K_4 = I_{16}$.

$$K_4\ vec(A) = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix} a_{11}\\0\\0\\0\\0\\a_{22}\\0\\0\\0\\0\\a_{33}\\0\\0\\0\\0\\a_{44} \end{bmatrix}
=
\begin{bmatrix} a_{11}\\0\\0\\0\\0\\a_{22}\\0\\0\\0\\0\\a_{33}\\0\\0\\0\\0\\a_{44} \end{bmatrix}
= vec\ (A^T)$$

b) In this case, if $a_{ii} = a_{jj}$ elements, then we have the permutation matrix takes the $(5i-4)^{th}$ element (a_{ii}) in $vec\ (A)$ into the $(5j-4)^{th}$ element (a_{jj}) in $vec\ (A^T)$, and takes the $(5j-4)^{th}$ element (a_{jj}) in $vec\ (A)$ into the $(5i-4)^{th}$ element (a_{ii}) in $vec\ (A^T)$. So we have the commutation matrix is $K_4 = P(5i-4 \quad 5j-4)$.

c) In this case, if $a_{kk} = a_{ll}$ elements, then we have the permutation matrix takes the $(5k-4)^{th}$ element (a_{kk}) in $vec\ (A)$ into the $(5l-4)^{th}$ element (a_{ll}) in $vec\ (A^T)$, and takes the $(5l-4)^{th}$ element (a_{ll}) in $vec\ (A)$ into the $(5k-4)^{th}$ element (a_{kk}) in $vec\ (A^T)$. So we have the commutation matrix is $K_4 = P(5k-4 \quad 5l-4)$.

d) In this case, if $a_{ii} = a_{jj}$ and $a_{kk} = a_{ll}$ elements, then we have the permutation matrix takes the $(5i-4)^{th}$ element (a_{ii}) in $vec\ (A)$ into the $(5j-4)^{th}$ element (a_{jj}) in $vec\ (A^T)$, takes the $(5j-4)^{th}$ element (a_{jj}) in $vec\ (A)$ into the $(5i-4)^{th}$ element (a_{ii}) in $vec\ (A^T)$, takes the $(5k-4)^{th}$ element (a_{kk}) in $vec\ (A)$ into the $(5l-4)^{th}$ element (a_{ll}) in $vec\ (A^T)$, and takes the $(5l-4)^{th}$ element (a_{ll}) in $vec\ (A)$ into the $(5k-4)^{th}$ element (a_{kk}) in $vec\ (A^T)$. So we have the commutation matrix is $K_4 = P(5i-4 \quad 5j-4)(5k-4 \quad 5l-4)$.

Thus, in a 4 x 4 diagonal matrix with $a_{ii} = a_{jj}$ and $a_{kk} = a_{ll}$ elements, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ are $I_{16}, P(5i-4 \quad 5j-4\ ), P(5k-4 \quad 5l-4), P(5i-4 \quad 5j-4)(5k-4 \quad 5l-4)$.

IV. CONCLUSION

From the result of research can be concluded that there are

1) For any diagonal matrix, one of the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ is identity matrix.
2) For the 4 x 4 diagonal matrix, which that two pairs of elements are located on the main diagonal with the same value, then the K_{mn} commutation matrix that satisfies the equation $K_{mn}\ vec\ (A) = vec\ (A^T)$ are $I_{16}, P(5i-4 \quad 5j-4\,), P(5k-4 \quad 5l-4), P(5i-4 \quad 5j-4)(5k-4 \quad 5l-4)$.

REFERENCES

[1]. K. M. Abadir and J. R. Magnus, Matrix Algebra (Econometric Exercises 1) (Cambridge University Press, New York, 2005).

[2]. C. Xu, L. He, and Z. Lin, Linear and Multilinear Algebra 68, 1721–1742 (2018).

[3]. H. Anton and C. Rorres, Elementary Linear Algebra Applications Version 11th Edition (John Willey & Sonc, Inc, Canada, 2014).

[4]. J. A. Gallian, Contemporary Abstract Algebra. Ninth Edition (Cencage Learning, Boston, 2017).

[5]. D. A. Harville, Matrix Algebra from a statistician's Perpective (Springer, New York, 2008).

[6]. R. Piziak and P. L. Odell, Matrix Theory: From Generalized Inverses to Jordan Form (Chapmann & Hall/CRC, New York, 2007).