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The Size Multipartite Ramsey Numbers $m_j(K_{1,n}, W_4)$ And $m_5(P_n, W_4)$

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Abstract – For given two any graph H and G, the size multipartite Ramsey number $m_j(H, G)$ is the smallest integer t such that for every factorization of graph Kj×t := F1 ⊕F2 so that F_1 contain H as a subgraph or F_2 contains G as a subgraph. In this paper, we determine $m_j(K_{1,n}, W_4)$ with $j = 4, 5$ and $m_5(P_n, W_4)$ for $n \geq 2$ where $K_{1,n}$ denotes a star on n+1 vertices, P_n denotes a path on n vertices, and W_4 denotes a wheel on 4 vertices.

Keywords – Paths, Size Multipartite Ramsey Numbers, Stars, Wheels

I. Introduction

Let *G*=(*V*,*E*) be a graph with the vertex-set *V*(*G*) and edge-set *E*(*G*). All graphs in this paper are finite and simple. Degree of a vertex v is the number of vertices adjacent to v, denoted $deg(v)$. So, the neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to v in *G*. The *minimum degree* and *maximum degree* of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any set $S \subseteq V(G)$, the *induced subgraph* *G*[*S*] of *G* by *S* is the maximal subgraph of *G* with the vertex-set *S*. If $e = uv \in E(G)$ then u is called *adjacent* to v. A graph *G* is said to be *factorable* into factors $G_1, \cdots, G_n$ if these factors are pairwise edge-disjoint and $\cup_{i=1}^{n} E(G_i) = E(G)$. If *G* is factored into $G_1, \cdots, G_n$, then $G = G_1 \oplus \cdots \oplus G_n$, which is called a *factorization* of *G*.

A *star* $K_{1,n}$ is the graph on $n + 1$ vertices with one vertex of degree *n*, called the *center*, and *n* vertices of degree 1. A *path* P_n is the graph on $n \geq 2$ vertices with two vertices of degree 1, and *n*-2 vertices on of degree 2. A *cycle* C_n is a 2-reguler connected graph. A wheel $W_n \cong C_n + \{x\}$ is a graph on *n* vertices with the hub x which adjacent to all vertices in C_n. Define aP_b is a path with a as initial vertex and b as terminal vertex.

The notion of size multipartite Ramsey numbers were introduced by Burger and Vuuren [3] in 2004, and Syafrizal *et al*. by considering the two factorization of a $K_{j\times t}$ by fixing the size *j* of the uniform multipartite sets. More precisely, For given two graphs G_1 and G_2, and integer $j \geq 2$, the size multipartite Ramsey numbers $m_j(G_1, G_{12}) = t$ is the smallest integer such that every factorization of graph $K_{j\times t} \coloneqq F_1 \oplus F_2$ satisfies the following condition: either F_1 contains G_1 as a subgraph or F_2 contains G_2 as a subgraph. Ramsey numbers of small paths versus cycle of three or four vertices have been studied by Syafrizal Sy [6].

There are only few results on the size multipartite Ramsey numbers m_j(G,H). In this paper, we consider a generalization of this concept by releasing completeness requirement in the forbidden graphs as follows. Syafrizal Sy [7] determined the exact values of the size multipartite Ramsey numbers of large path versus wheel on five vertices. The size multipartite Ramsey numbers $m_j(K_{1,m}, P_n)$ and $m_2(K_{1,m}, C_n)$ was studied by Lusiani *et al*. [5]. The size multipartite Ramsey numbers $m_j(K_{1,t}, P_3) = n$ for

$j, n \geq 3$ studied by Baqi *et al.* [1]. Furthermore, Baskoro *et al.* [2] studied size multipartite Ramsey numbers for star and cycle $m_j(s_m, C_n)$ for $3 \leq n \leq j$ and $m \geq 3$. Effendi *et al.* [4] determined size multipartite Ramsey numbers for combination path and wheel on four vertices $m_3(P_n, W_4)$ for $n \geq 3$. The aim of this paper is determined $m_j(K_{1,n}, W_4)$ with $j = 4{,}5$ and $m_5(P_n, W_4)$ for $n \geq 2$. In this note, we prove the following theorem.

II. Size Ramsey Numbers Related To $K_{1,n}$ And W_4

We will determine the size multipartite Ramsey numbers for star versus wheel on 4 vertices as the following theorem.

Theorem 3.1. For positive integer $n \geq 2$,

$$m_4(K_{1,n}, W_4) = \begin{cases} 2 & for\ n = 2, \\ 3 & for\ n = 3, \\ \left\lceil \frac{n-1}{3} \right\rceil + 2 & for\ n \geq 4. \end{cases}$$

Proof. We consider three cases as follow.

Case 1. For $n = 2$.

The first, we determine the lower bound $m_4(K_{1,2}, W_4) \geq 2$. Let $F_1 \oplus F_2$ be the factorization of graph $F = K_{4\times1}$ such that F_1 contains no $K_{1,2}$ as subgraph. We assume that F_1 contains a perfect matching $M = \{a_{11}a_{21}, a_{31}a_{41}\}$. Let $a_{21} \in V_2$ be a hub of wheel W_4 and let $N(x)$ be the set of vertices adjacent to x in F_1, then $|V(F_1)\backslash(V_2 \cup N(x))| < |V(C_4)|$. Clearly that F_2 contains no W_4 as a subgraph. Therefore, $m_4(K_{1,n}, W_4) \geq 2$.

Next, we will determined of upper bound $m_4(K_{1,2}, W_4) \leq 2$. Let $G_1 \oplus G_2$ be any the factorization of $G = K_{4\times2}$ such that G_1 contains no $K_{1,2}$ as a subgraph. We will show that G_2 contain W_4 as a subgraph. Let $V_i = \{a_{i1}, a_{i2}\}$ for $i = 1, 2, 3, 4$ be the partite set of G. Since G_1 contains no $K_{1,2}$ then $\Delta(G_1) \leq 1$. Assume that G_1 contains perfect matching $M^1 = \{a_{11}a_{41}, a_{12}a_{21}, a_{22}a_{32}, a_{31}a_{42}\}$. Suppose partite V_4 contain vertex $x = a_{42}$ as the center of W_4. Hence, these all vertices a_{11}, a_{22}, a_{12}, and a_{32} will form cycle on four vertices where the set of vertices is $C_4 := a_{11}, a_{22}, a_{12}, a_{32}, a_{11}$ in G_2. As a consequence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_4(K_{1,n}, W_4) \leq 2$.

Case 2. For $n = 3$.

We will show first that of the lower bound $m_4(K_{1,3}, W_4) \geq 3$. Let $F_1 \oplus F_2$ be a factorization of $F = K_{4\times2}$ such that F_1 contains no $K_{1,3}$ as a subgraph. Let $V_i = \{a_{i1}, a_{i2}\}$ for $i = 1, 2, 3, 4$ be the partite set of F. Thus, F_1 contains no $K_{1,3}$ as a subgraph. Assume that $F_1 = 2C_4$ with $V(C_4^1) = \{a_{11}, a_{22}, a_{31}, a_{41}, a_{11}\}$ and $V(C_4^2) = \{a_{12}, a_{21}, a_{32}, a_{42}, a_{12}\}$. Take vertex $x \in V_1$ as a hub of wheel W_4 and $N(x)$ is the set of vertices adjacent to x in F_1, so that $|V(F)\backslash(V_1 \cup N(x))| < |E(C_4)|$. Thus, F_2 contains no W_4 as a subgraph. Therefore, $m_4(K_{1,3}, W_4) \geq 3$.

Next, to show the upper bound of $m_4(K_{1,3}, W_4) \leq 3$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{4\times3}$ such that G_1 contains no $K_{1,3}$ as a subgraph. We will show that G_2contain W_4 as a subgraph. Let $V_i = \{a_{i1}, a_{i2}, a_{i3}\}$ for $i = 1, 2, 3, 4$ be the partite set of G. Since G_1contains no $K_{1,3}$, then $\Delta(G_1) \leq 2$. Assume that G_1 contain $C_{12}$with $V(C_{12}) = \{a_{11}, a_{23}, a_{33}, a_{13}, a_{21}, a_{42}, a_{32}, a_{12}, a_{41}, a_{22}, a_{43}, a_{31}, a_{11}\}$. Suppose vertex $x = a_{21}$ as a hub of wheel W_4 in G_2. Let $N(x)$ be the set of vertices adjacent to x in G_1, such that $G_2[V(G)\backslash(V_2 \cup N(x))]$ has 7 vertices. Thus, we have the set of vertices of cycle $C_4 := a_{12}, a_{33}, a_{41}, a_{31}, a_{12}$ in G_2. So, we have wheel $W_4 := C_4 + \{x\}$ as a subgraph in G_2. Therefore, $m_4(K_{1,3}, W_4) \leq 3$.

Case 3. For $n \geq 4$.

Suppose $p = \left\lceil \frac{n-1}{3} \right\rceil + 2$. We will show first the lower bound $m_4(K_{1,n}, W_4) \geq p$. Let $F_1 \oplus F_2$ be the any factorization of $F = K_{4\times(p-1)}$ such that F_2 contains no W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, \cdots, p-1$ be the partite sets in F. Since F_2 contains no W_4 as a subgraph, then the maximal degree is 3 for every $a_{ij} \in V(F_1)$. Suppose vertex

$x \in V_i$ is a center of $K_{1,n}$, then clearly that F_1 contains no $K_{1,n}$ as a subgraph. Since $V(F) = 3\left\lfloor\frac{n-1}{3}\right\rfloor$ then $V(F_1) = 3\left\lfloor\frac{n-1}{3}\right\rfloor - 3 < V(K_{1,n})$. Therefore, $m_4(K_{1,n}, W_4) \geq p$.

Next, to show the upper bound of $m_4(K_{1,n}, W_4) \leq p$. Let $G_1 \oplus G_2$ be the factorization of $G = K_{4\times p}$ such that G_1 contains no $K_{1,n}$ as a subgraph. We will to show that G_2 contain W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ be the partite set of G for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, \cdots, p$. Since G_1 contains no $K_{1,n}$, then $\Delta(G_1) \leq n-1$ in G_1. Suppose there is exist a one vertex $x \in V_i$ as a hub of W_4 in G_1. Let $N(x)$ be the set of all vertices adjacent to x in G_1, then $G_2[V(G)\backslash(V_i \cup N(x))]$ has $3p-(n-1)$ vertices, and minimum degree $\delta(G_2[V(G)\backslash V_i \cup N(x)]) \geq 3p-(n-1)$. Since there exist at least four vertices, namely a, b, c, and d will contain cycle C_4 in G_2 t at least, then G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_4(K_{1,n}, W_4) \leq p$.

Theorem 3.2. For positive integer $n \geq 2$,

$$m_5(K_{1,n}, W_4) = \begin{cases} 1 & for\, n = 2, \\ \left\lfloor\frac{n}{4}\right\rfloor + 1 & for\, n = 8k+2, k \in \mathbb{Z}^+, \\ \left\lfloor\frac{n+2}{4}\right\rfloor + 1 & for\ n\ others. \end{cases}$$

Proof. We consider three cases as follow.

Case 1. For $n = 2$.

We will show first the lower bound of $m_5(K_{1,2}, W_4) \geq 1$. Let $F_1 \oplus F_2$ be the any factorization of $F = K_{5\times(1-1)}$. Clearly that $m_5(K_{1,2}, W_4) \geq 1$.

Next, to show the upper bound of $m_5(K_{1,2}, W_4) \leq 1$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times 1}$ such that G_1 contains no $K_{1,2}$ as a subgraph. We will show that G_2 contain W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ for $i = 1, 2, 3, 4, 5$ be the partite set in G. Since G_1 contains no $K_{1,2}$, then $\Delta(G_1) \leq 1$. Assume that G_1 contain a matching $M^2 := \{a_{21}a_{31}, a_{41}a_{11}\}$ such that there exist one vertex $x = a_{11}$ as a center of W_4. Since $|V(G_1)\backslash x| = 4$, then there the vertex set of cycle $C_4 := a_{21}, a_{51}, a_{31}, a_{41}, a_{21}$ in G_2. So, G_2 will contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(K_{1,2}, W_4) \leq 1$.

Case 2. For $n = 8k+2, k \in \mathbb{Z}^+$.

We will show first the lower bound of $m_5(K_{1,n}, W_4) \geq \left\lfloor\frac{n}{4}\right\rfloor + 1$ for $n = 8k+2, k \in \mathbb{Z}^+$. Let $F_1 \oplus F_2$ be the any factorization of $F = K_{5\times\left\lfloor\frac{n}{4}\right\rfloor}$ such that F_2contains no W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ be the partite set of F for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, \cdots, \left\lfloor\frac{n}{4}\right\rfloor$. Since F_2 contains no W_4 as a subgraph, then maximal degree is 3 for every $a_{ij} \in V(F)$. Suppose $x \in V_i$ is the center of $K_{1,n}$. Since $deg(F) = 4\left\lfloor\frac{n}{4}\right\rfloor$, then $deg(F_1) = 4\left\lfloor\frac{n}{4}\right\rfloor - 3 < deg(K_{1,n})$. As a consequence, F_1 contains no $K_{1,n}$ as a subgraph. Therefore, $m_5(K_{1,n}, W_4) \geq \left\lfloor\frac{n}{4}\right\rfloor + 1$ for $n = 8k+2, k \in \mathbb{Z}^+$.

Next, to show the upper bound of $m_5(K_{1,n}, W_4) \leq \left\lfloor\frac{n}{4}\right\rfloor + 1$ for $n = 8k+2, k \in \mathbb{Z}^+$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times\left\lfloor\frac{n}{4}\right\rfloor+1}$ such that G_1 contains no $K_{1,n}$ as a subgraph. We will show that G_2 contain W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ be partite set of G for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, \cdots, \left\lfloor\frac{n}{4}\right\rfloor + 1$. Since G_1contains no $K_{1,n}$ as a subgraph, then $\Delta(G_1) \leq n-1$ for $n = 8k+2$, such that the partite V_i contain one vertex $x = a_{ij}$ with $deg(x) = n-2$ as a hub of W_4 in G_2. Let $N(x)$ be vertex set adjacent to x in G_1. Since $G_2[V(G)\backslash (V_i \cup N(x))]$ is $4\left\lfloor\frac{n}{4}\right\rfloor + 1 - (n-2)$ vertices. As a consequence, there are four vertices a, b, c, and d will form cycle C_4 in G_2, such that G_2 contain $W_4 := C_4 + \{x\}$. Therefore, $m_5(K_{1,n}, W_4) \geq \left\lfloor\frac{n}{4}\right\rfloor + 1$.

Case 3 . For other n.

We will show first the lower bound of $m_5(K_{1,n}, W_4) \geq \left\lfloor \frac{n+2}{4} \right\rfloor + 1$ for other n. Let $F_1 \oplus F_2$ be the any factorization of $F = K_{5\times\left\lfloor \frac{n+2}{4} \right\rfloor}$ such that, F_2 contains no W_4 as a subgraph. Let $V_i = \{a_{ij}\}$ be the partite set of F for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, \cdots, \left\lfloor \frac{n+2}{4} \right\rfloor$. Since F_2 contains no W_4 as a subgraph, then the maximal degree for $a_{ij} \in V(F)$ is 3. Since $x \in V_i$ is a center of $K_{1,n}$ and $deg(F) = 4\left\lfloor \frac{n+2}{4} \right\rfloor$, then $deg(F_1) = 4\left\lfloor \frac{n+2}{4} \right\rfloor - 3 < deg(K_{1,n})$. Clearly that, F_1 contains no $K_{1,n}$ for other n as a subgraph. Therefore, $m_5(K_{1,n}, W_4) \geq \left\lfloor \frac{n+2}{4} \right\rfloor + 1$ for other n.

Next, to show the upper bound of $m_5(K_{1,n}, W_4) \leq \left\lfloor \frac{n+2}{4} \right\rfloor + 1$ for other n. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times\left\lfloor \frac{n+2}{4} \right\rfloor+1}$ such that G_1 contains no $K_{1,2}$ as a subgraph. To show that G_2 contain W_4 as a subgraph, suppose $V_i = \{a_{ij}\}$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, \cdots, \left\lfloor \frac{n+2}{4} \right\rfloor + 1$ is a partite set of G. Since G_1contains no $K_{1,n}$ as a subgraph, then $\Delta(G_1) \leq n - 1$ for other n. Suppose $x \in V_i$ is a hub of W_4. Since $N(x)$ is the set of vertices adjacent to x in G_1, then $G_2[V(G)\backslash(V_i \cup N(x))]$ has $4\left(\left\lfloor \frac{n+2}{4} \right\rfloor + 1\right) - (n-1)$ vertices and $\delta(G_2) \geq 4\left(\left\lfloor \frac{n+2}{4} \right\rfloor + 1\right) - (n-1)$. As a consequence, all these vertices a, b, c, and d will form cycle C_4 in G_2, such that G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(K_{1,n}, W_4) \leq \left\lfloor \frac{n+2}{4} \right\rfloor + 1$.

III. Size Ramsey Numbers Related To P_n And W_4

We will determine the size multipartite Ramsey numbers for path versus wheel on 4 vertices as the following theorem.

Theorem 4.1. For positive integer $n \geq 2$,

$$m_5(P_n, W_4) = \begin{cases} 1 & for\ 2 \leq n \leq 3, \\ 2 & for\ 4 \leq n \leq 5, \\ \left\lfloor \frac{2n+3}{5} \right\rfloor & for\ \ n \geq 6. \end{cases}$$

Proof. We consider three cases as follow.

Case 1. For $2 \leq n \leq 3$.

We will show first that the lower bound of $m_5(P_n, W_4) \geq 1$. Let $F_1 \oplus F_2$ be the any factorization of $F = K_{5\times(1-1)}$. Clearly that, $m_5(P_n, W_4) \geq 1$ for $2 \leq n \leq 3$.

To show the upper bound of $m_5(P_2, W_4) \leq 1$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times1}$ such that G_1 contains no P_2 as a subgraph. We will show that G_2 contain W_4 as a subgraph. Suppose $V_i = \{a_{i1}\}$ for $i = 1, 2, 3, 4, 5$ is a partite set of G. Since G_1contains no P_2 as a subgraph, then $G_1$is contain independent vertices. Clearly that, $|V(G_1)\backslash V_i| = 4$, thus G_2 contain cycle $C_4 := a_{11}\, a_{41}\, a_{31}\, a_{51}\, a_{11}$, so that G_2 contain W_4 as a subgraph. Therefore, $m_5(P_2, W_4) \leq 1$.

Next, to show the upper bound of $m_5(P_3, W_4) \leq 1$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times1}$ such that G_1 contains no P_3 as a subgraph. We will show that G_2 contain W_4 as a subgraph. Suppose $V_i = \{a_{i1}\}$ is a partite set in G for $i = 1, 2, 3, 4, 5$. Since G_1contains no P_3 as a subgraph, we assume G_1 contain a matching $M^2 = \{a_{11}a_{31}, a_{51}a_{41}\}$. So, there is exist one vertex, namely $x = a_{21}$, as a hub of W_4. Since $|V(G_1)\backslash V_i| = 4$, then there exist $C_4 := a_{11}\, a_{41}a_{31}a_{51}\, a_{11}$ in G_2. Clearly that, vertex x adjacent to all vertices in G_2. Hence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_3, W_4) \leq 1$.

Case 2. For $4 \leq n \leq 5$.

We will show first that the lower bound of $m_5(P_n, W_4) \geq 2$ for $4 \leq n \leq 5$. Let $F \oplus F_2$ be the any factorization of $F = K_{5\times(2-1)}$ such that F_1 contain no P_n as a subgraph. Suppose $V_i = \{a_{i1}\}$ is the partite set in F for $i = 1, 2, 3, 4, 5$. Since F_1 contains no P_n as a subgraph and $|V(F_2)\backslash V_i| < |V(C_4)|$, then clearly that F_2 contains no W_4 as a subgraph. Therefore, $m_5(P_n, W_4) \geq 2$.

Next, to show the upper bound of $m_5(P_4, W_4) \leq 2$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times2}$ such that G_1 contain no P_4 as a subgraph. To show G_2 contain W_4, we consider two the following.

Case 2.1. If $G_1 = 3K_3 \cup P_4$.

Let $V_i = \{a_{i1}, a_{i2}\}$ be the partite set of G for $i = 1,2,3,4,5$. Suppose $V(K_3^1) = \{a_{11}, a_{22}, a_{31}, a_{11}\}$, $V(K_3^2) = \{a_{12}, a_{21}, a_{52}, a_{12}\}$, $V(K_3^3) = \{a_{32}, a_{41}, a_{51}, a_{32}\}$ is a graph $3K_3$ and $V(P_1) = a_{41}$ in G_1. Since G_1 contain no P_4 as a subgraph, then vertex $V(P_1) = a_{41}$ no adjacent to every vertices in G_2 such that vertex a_{41} is hub of W_4, such that C_4 $:= a_{12}, a_{22}, a_{15}, a_{31}, a_{12}$ in. Thus, G_2 contain $W_4 := C_4 + \{x\}$. Therefore $m_5(P_4, W_4) \leq 2$

Case 2.2. If $G_1 \neq 3K_3 \cup P_1$.

Since G_1 contains no $3K_3 \cup P_1$, then there is exist x one vertex x with $deg(x) \leq 1$. Suppose $A = V(K_{5\times 2}) \setminus (V_i \cup N(x))$, such that $|V(A)| \geq 7$. Since $P = aP_b$ is longest path in A, then $xa, xb \notin E(G_1)$. Next, Suppose $I = V(A) \setminus (V_a \cup V_b)$ is the subset of induce subgraph $G_1[A]$. Since $|V(I)| \geq 3$, then there exist at leas two vertices, namely c and d, where $c, d \in (G_1[I])$. Since $ab, bc, cd, da \notin E(G_1)$ and $xd, xc \notin E(G_1)$, then these all vertices a, b, c and d will form C_4 in G_2 such that G_2 contain W_4 $:= C_4 + \{x\}$. Therefore, $m_5(P_4, W_4) \leq 2$.

Next, to show the upper bound of $m_5(P_5, W_4) \leq 2$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times 2}$ such that G_1 contains no P_5 as a subgraph. We will show that G_2 contain W_4 as a subgraph whit two the following cases.

Case 2.3. If $G_1 = 2K_4 \cup P_2$.

Suppose $V_i = \{a_{i1}, a_{i2}\}$ with $i = 1,2,3,4,5$ is a partite set in G. Since G_1 contain $2K_4 \cup P_2$ as a subgraph, then we may assume that there exist a subgraph $2K_4 \cup P_2$ with $V(K_4^1) = \{a_{11}, a_{22}, a_{32}, a_{51}\}$, $V(K_4^2) = \{a_{12}, a_{21}, a_{41}, a_{52}\}$ contain $2K_4$, and $V(P_2) = \{a_{31}, a_{42}\}$ in G_1, such that a_{31} will form a wheel W_4 whit x a as hub. A consequence, all these vertices $a_{11}, a_{21}, a_{51}, a_{41}$ also no adjacent to all vertex in G_1, such that will form cycle C_4 in G_2. Hence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_5, W_4) \leq 2$.

Case 2.4. If $G_1 \neq 2K_4 \cup P_2$.

Since G_1 contains no $2K_4 \cup P_2$, then there exist one vertex x with $deg(x) \leq 2$. Since $B = V(K_{5\times 2}) \setminus (V_x \cup N(x))$ then $|V(B)| \geq 6$. Since $P = P_b$ is the longest of B, then $xa, xb \notin E(G_1)$. Next, since $L = V(B) \setminus (V_a \cup V_b)$ where $V(B)$ is a subset of B which induced subgraph by G_1, so that $|V(L)| \geq 2$, then there are exist at least two vertices, namely c and d, with $c, d \in (G_1[L])$. A a consequence, since these all edges $ab, bc, cd, da \notin E(G_1)$ and $xc, xd \notin E(G_1)$ then these all vertices a, b, c and d will form a cycle C_4 in G_2, such that G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_5, W_4) \leq 2$.

Case 3. For $n \geq 6$.

Suppose $s = \left\lceil \frac{2n+3}{5} \right\rceil$. We will show the upper bound of $m_5(P_n, W_4) \geq 2$ for ≥ 6 . Let $F \oplus F_2$ be the any factorization of $F = K_{5\times(s-1)}$ such that F_1 consist of two partitions, namely J_1 and J_2, with J_1 contain complete multipartite graph and J_2 is complement of J_1. Since $|V(J_1)| = n - 1 < n$ and $|V(J_2)| = 5\left\lceil \frac{2n+3}{5} \right\rceil - (n-1) < n$, then F_1 contains no P_n as a subgraph. Since F_1 consist of two partitions, then clearly that F_2 contains no W_4 as a subgraph. Therefore, $m_5(P_n, W_4) \geq s$.

Next, to show the upper bound of $m_5(P_n, W_4) \leq s$. Let $G_1 \oplus G_2$ be the any factorization of $G = K_{5\times s}$ such that G_1contains no P_n as a subgraph for $n \geq 6$. Assume that G_1 contain longest path, namely $P = aP_b$. We will show that G_2 contain W_4 as a subgraph so that, we consider four possibilities.

Case 3.1. If $V_a = V_b$ and $V_c = V_d$.

Let P be the set of vertices in G_1, then $A = V(K_{5\times s}) \setminus (V_b \cup N(P))$. Next, Suppose $Q = cQ_d$ is the longest path in G_1 which induced $G[A]$ and $B = V(A) \setminus (V_c \cup N(Q))$. Since $ac, cb, bd, da \notin E(G_1)$, then the all vertices a, b, c and d ca be form a cycle C_4 in G_2. Furthermore, since $5\left\lceil \frac{2n+3}{5} \right\rceil - 2(n-1)$ with $\delta(G_2) \geq 1$, so that there is one vertex x in $V(G_2) \setminus (V_b \cup V_c)$ such that x adjacent to all edges $ac, cb, bd, da \in E(G_2)$. As a consequence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_n, W_4) \leq s$.

Case 3.2. If $V_a = V_b$ and $V_c \neq V_d$.

Let P be the set of vertices in G_1, then $A = V(K_{5\times s})\backslash(V_b \cup N(P))$. Next, Suppose $Q = cQ_d$ is the longest path in G_1 which induced $G[A]$ and $B = V(A)\backslash(V_c \cup V_d \cup N(Q))$. Since $ac, cb, bd, da \notin E(G_1)$, then all edges ac, cb,bd, da will form C_4 in G_2. So, since $5\left\lfloor\frac{2n+3}{5}\right\rfloor - 2(n-1)$ with $\delta(G_2) \geq 1$, such that there is exist one vertex x in $G_2(V_b \cup V_c \cup V_d)$ which this implies of x adjacent to all edges $ad, db, bc, ca \in E(G_2)$. Hence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_n, W_4) \leq s$.

Case 3.3. If $V_a \neq V_b$ and $V_c = V_d$

Let P be the set of vertices in G_1, then $A = V(K_{5\times s})\backslash(V_a \cup V_b \cup N(P))$. Next, Suppose $Q = cQ_d$ is the longest path in G_1 which induced $G[A]$ and $B = V(A)\backslash(V_d \cup N(Q))$. Since $ac, cb, bd, da \notin E(G_1)$, then the all these vertices a, b, c, d will be form cycle C_4 in G_2. Since $5\left\lceil\frac{2n+3}{5}\right\rceil - 2(n-1)$ with $\delta(G_2) \geq 1$, such that there is exist one vertex x in $G_2\backslash(V_a \cup V_b \cup V_d)$ which this implies of x adjacent to all sisi $ac, cb, bd, da \in (G_2)$. Thus, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_n, W_4) \leq s$.

Case 3.4. If $V_a \neq V_b$ and $V_c \neq V_d$.

Let P be the set of vertices in G_1, then $A = V(K_{5\times s})\backslash(V_a \cup V_b \cup N(P))$. Next, Suppose $Q = cQ_d$ is the longest path in G_1 which induced $G[A]$ and $B = V(A)\backslash\ (V_c \cup V_d \cup N(Q))$. Since $ac, cb, bd, da \notin E(G_1)$, such that the all these vertices a, b, c, and d will be form cycle C_4 in G_2. Since $5\left\lfloor\frac{2n+3}{5}\right\rceil - 2(n-1)$ with $\delta(G_2) \geq 1$, such that there is one vertex x in $G_2\backslash(V_a \cup V_b \cup V_c \cup V_d)$ which this implies of x adjacent to all vertices $ac, cb, bd, da \in E(G_2)$. Hence, G_2 contain $W_4 := C_4 + \{x\}$ as a subgraph. Therefore, $m_5(P_n, W_4) \leq s$.

IV. CONCLUSIONS

In this paper, we obtain the size multipartite Ramsey numbers for $m_j(K_{1,n}, W_4)$ for $j = 4{,}5$ and $m_5(P_n, W_4)$ with $n \geq 2$.

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