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# On The Solutions of The Image Processing Problems Using Total Variational-Based Diffusion and Digital-Discrete Method

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**Abstract – In this paper, we combine Total Variational(TV)-based diffusion and Digital-Discrete Method for image processing problems. We propose an algorithm based on partial differential equation of TV regularization model, the finite difference method and gradually varied function (GVF). MATLAB package program is used to analyze images by using our algorithm. We see that this new algortihm is effective and simple to use in image processing.**

**Keywords – Total Variational(TV)-based diffusion, digital-discrete method, gradually varied function (GVF), finite difference method, image processing.**

## I. Introduction

Image processing problems are one of the most interesting topics of today. Tools such as mobile phones, computers, televisions, videos, cameras and magnetic resonance, which are indispensable in the modern world, need constant innovations for people's comfort.[1]

Increasing the image quality is an issue that attracts the attention of mathematicians like everyone else. As mathematicians, we thought about what can be done about the solutions of this problem. We created an algorithm using digital-discrete method, finite difference method, Total Variational(TV)-based diffusion MATLAB package program and total variational based diffusion to improve image quality.[1]

Especially in recent years, the use of partial differential equations(PDEs) in imaging processes has increased. Various models of partial differential equations have been used. These models are evolutionary PDEs [2], geometric and variational PDEs [3], fourth-order PDEs [4,5] and adaptive PDEs [6]. Some newer methods include semi-implicit finite volume scheme [7], additive operator splitting method [8], topological gradient approach [9], a restarted iterative homotopy analysis method [10] , hybrid analog-digital method [11]. In this study, we apply digital-discrete method for Total Variational(TV)-based diffusion in image processing.

## II. Gradually Varied Function (GVF) and its Algortihm [1,12]

### 2.1. GVF

GVF was introduced and developed by Chen in digital and discrete spaces [1,12,13,14].

$A_1, A_{2,\dots,}A_m \in \mathbb{R}$, $A_1 < A_2 < \cdots < A_m$ and $p, q \in \sum_2$ are given.

Let $f: \sum_2(or\ another\ discrete\ space) \to \{A_1, A_{2,\dots,}A_m\}$ be a function.

If $f(p) = A_i$ and $f(q) = A_j$ , then level difference(LD) is | i-j | between *f(p)* and *f(q)*. The GVF can be defined as follows.

***Definition 2.1.1.*** **[1,12,13,14]:** For any *p, q* close pair in $\sum_2$,

If $f(p) = A_i$ and $f(q) = A_{i-1}, A_i$ *or* $A_{i+1}$ , then $f$ is GVF on $p$ and $q$.

***Definition 2.1.2.*** **[1,12,13,14]:** If $f$ is GVF on any *p, q* close pair in $\sum_2$, then $f$ is GVF.

***Theorem 2.1.1.*** **[1,12,13,14]:** Let *p*, *q* be points in J, if a gradually varied interpolation exists, then length of the shortest path in D between *p* and *q* is not less than level difference between *f(p)* and *f(q)*.

**2.2. GVF Algorithm [1,12,15]:**

*J* is a non-empty subset of *D*. $f_J$ is a function defined on *J*.

**Step 1:** We test all *p* and $p'$ points in *J*. If $d(p, p') \geq LD(p, p')$ is not satisfied, then there is no GVF. Take $D_0 \leftarrow J$.

**Step 2:** We take *x* from $D - D_0$ where *x* has an adjacent vertex *r* in $D_0$. Suppose $f_D(r) = A_i$.

**Step 3:** We take $f_D(x) = f_D(r) = A_i$. We test *x* against every vertex *p* in $D_0$: If there is a $p \in D_0$ when $d(x, p) < LD(x, p)$, change $f_D(x)$ to $A_{i-1}$ when $f_D(p) < A_i$ or change $f_D(x)$ to $A_{i+1}$ when $f_D(p) > A_i$.

**Step 4:** Let $D_0 \leftarrow D_0 \cup \{x\}$.

**Step 5:** We repeat 2–4 until $D_0 = D$.

## III. DIGITAL-DISCRETE METHOD FOR TOTAL VARIATIONAL-BASED DIFFUSION

**3.1. Total Variational-Based Diffusion [16,17]:**

TV based filtering was introduced by Rudin, Osher and Fatemi based on the minimization of the energy functional [18]. The Total Variation (TV) model is generally defined as follows [16,17,18]:

$$E(u) = \int_\Omega \left[\frac{1}{2}(u-f)^2 + \alpha|\nabla u|\right] dx \qquad (3.1.1)$$

- $\Omega \subset \mathbb{R}$ is bounded, connected, open subset representing the image domain.
- $f$ is an image on $\Omega$.
- $u$ is a restored version of $g$.
- $\alpha > 0$ is a scalar.

The gradient descent equation for (3.1.1) is defined as follows [16,17,18]:

$$\frac{\partial u}{\partial t} = \nabla\left(\frac{\nabla u}{|\nabla u|}\right) - \frac{1}{\alpha}(u-f)$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \qquad (3.1.2)$$

The value of $\alpha$ indicates the relative importance of the truth term. Smoothness level depends on this scalar parameter. The observed $f$ image is assumed to be distorted by the addition of a zero-mean Gaussian blur, and the variance is known as $\alpha^2$ [18]. Therefore, in order to restore a given image, they propose to solve the following constrained optimization problem [18]:

$$\min_u \int_\Omega |\nabla u| dx \qquad \text{subject to}$$

$$\int_\Omega (u-f)^2 dx = \alpha^2. \qquad (3.1.3)$$

When the TV regularization is defined as a constrained optimization problem, $\frac{1}{\alpha}$ can be thought of as a Lagrange multiplier, which we can calculate by adding the given constraint into the calculation.

TV regularization can be considered common with nonlinear diffusion filtering, which is called TV flow [19,20].

If we neglect the truth term in equation (3.1.2), the partial differential equation becomes

$$\frac{\partial u}{\partial t} = \nabla(g(|\nabla u|)\nabla u) \tag{3.1.4}$$

with $u^0 = f$ and the diffusivity function $g(|\nabla u|) = \frac{1}{|\nabla u|}$. Compared to Perona-Malik diffusion, this diffusion function appears to have no additional contrast parameters.

The main purpose in Total Variation is the minimization of the energy functional [16,17]

$$E(u) = \int_{\Omega} |\nabla u| d\Omega \tag{3.1.5}$$

subject to the two constraints $\int_{\Omega} u d\Omega = \int_{\Omega} f d\Omega$ and $|u - f| = \alpha^2$. Depending on these constraints, minimization of the energy functional leads us to the following Euler-Lagrange equation:

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) - \frac{1}{\alpha}(u - f) = 0 \tag{3.1.6}$$

$$u_t = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}} - \frac{1}{\alpha}(u - f) \tag{3.1.7}$$

### 3.2. The finite difference method for Total Variational-Based Diffusion [16,17,21]:

Let's make equation (3.1.7) discrete. We can divide the ranges

$$\begin{aligned} x_i &= ih_1, \quad i = 1,2, \dots, N \\ y_j &= jh_2, \quad j = 1,2, \dots, M \\ t_k &= k\Delta t, \quad k = 1,2, \dots, n \end{aligned} \tag{3.2.1}$$

We take $h_1 = h_2 = 1$.

The discrete form of (3.1.7) becomes as follows

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = \left(\left(\frac{u_{i+1j}^k - u_{i-1,j}^k}{2}\right)^2 + \left(\frac{u_{i,j+1}^k - u_{i,j-1}^k}{2}\right)^2\right)^{-\frac{3}{2}} \cdot \left[\left(u_{i+1j}^k - 2u_{i,j}^k + u_{i-1,j}^k\right) \cdot \left(\frac{u_{i,j+1}^k - u_{i,j-1}^k}{2}\right)^2 - \frac{1}{8}\left(u_{i+1j}^k - u_{i-1,j}^k\right)\left(u_{i,j+1}^k - u_{i,j-1}^k\right)\left(u_{i+1,j+1}^k - u_{i+1,j-1}^k - u_{i-1,j+1}^k + u_{i-1,j-1}^k\right) + \left(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k\right)\left(\frac{u_{i+1j}^k - u_{i-1,j}^k}{2}\right)^2\right] - \frac{1}{\alpha}\left(u_{i,j}^k - f_{i,j}\right)$$

(3.2.2)

### 3.3. On the solutions of the Image Processing Problems using Total Variational-Based Diffusion and Digital-Discrete Method

We discretized the Total Variational-Based Diffusion by using the finite difference method in Chapter 3.2. We apply digital-discrete adaptation in (k+1) so that $u_{i,j}^{k+1} \leftarrow \left(u_{i,j}^{k+1} + gvf(i,j)/2\right)$. We combine gradually varied function, finite difference method and Total Variational-Based Diffusion for our algorithm [12,16,17,21].

**Algorithm (Digital-Discrete method for Total Variational-Based Diffusion in image processing) [1,12]**

**Step 1:** Upload the main points. Data points with observation values are loaded.

**Step 2:** Arrange the points in the lattice space and identify the solution..

**Step 3:** Expand the function according to Theorem 2.1.1 using the local Lipschitz condition. GVFs are obtained. (We apply GVF Algorithm)

**Step 4:** Start to get the lattice points with the iteration of Total Variational-Based Diffusion that we obtained by the finite difference method:

$$\frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t}=\left(\left(\frac{u_{i+1j}^{k}-u_{i-1,j}^{k}}{2}\right)^{2}+\left(\frac{u_{i,j+1}^{k}-u_{i,j-1}^{k}}{2}\right)^{2}\right)^{-\frac{3}{2}}.\Bigg[\left(u_{i+1j}^{k}-2u_{i,j}^{k}+u_{i-1,j}^{k}\right).\left(\frac{u_{i,j+1}^{k}-u_{i,j-1}^{k}}{2}\right)^{2}-\frac{1}{8}\left(u_{i+1j}^{k}-u_{i-1,j}^{k}\right)\left(u_{i,j+1}^{k}-u_{i,j-1}^{k}\right)\left(u_{i+1,j+1}^{k}-u_{i+1,j-1}^{k}-u_{i-1,j+1}^{k}+u_{i-1,j-1}^{k}\right)+\left(u_{i,j+1}^{k}-2u_{i,j}^{k}+u_{i,j-1}^{k}\right)\left(\frac{u_{i+1j}^{k}-u_{i-1,j}^{k}}{2}\right)^{2}\Bigg]-\frac{1}{\alpha}\left(u_{i,j}^{k}-f_{i,j}\right)$$

**Step 5:** Restore the current values obtained in step 4 using a GVF:

We continue to apply digital-discrete adaptation in (k+1) so that

$$u_{i,j}^{k+1}\leftarrow\left(u_{i,j}^{k+1}+gvf(i,j)/2\right)$$

**Step 6:** Perform image processing analysis with the help of the MATLAB-R2020b package program using our algorithm.

We obtained images for different *t* values in Figure 3.3.1, Figure 3.3.2 and Figure 3.3.3 by using our algorithm.

**Original Image**

T=50

T=100


**Figure 7.3.1.**

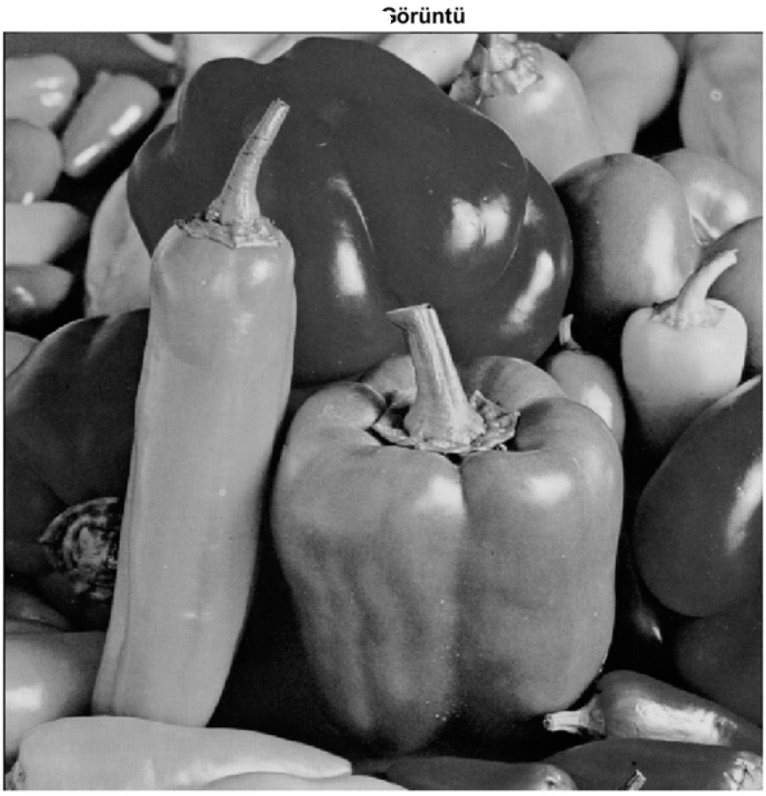
**Original Image**

T=50

T=100


**Figure 7.3.2.**

**Original Image**



T=50

T=100

T=200

**Figure 7.3.3.**

## IV. Conclusion

In this paper, we applied Digital-Discrete method for Total Variational-Based Diffusion to obtain a new and effective algorithm in image processing. The solution algorithm in this study

is simpler than other algorithms. Also our algorithm is easier to apply to package programs such as MATLAB. We think that digital-discrete method and its adaptation to nonlinear partial differential equations in imaging systems can be enlarged to other problems like biomedical systems.

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