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# *On One Conditionally Correct Cauchy Problem*

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**Abstract – This paper considers Cauchy problems for differential equations with operator coefficients in real Hilbert spaces. As is known (see, eg, [1,2]), such problems are usually incorrect in the classical sense. Therefore, we will examine them for conditional correctness.**

**Keywords – Scalar argument, algebra,** $a_{kl}(t)-$ **scalar functions.**

## I. Introduction

Consider the following Cauchy problem in Hilbert space H

$$\frac{d^n u(t)}{dt^n} + \sum_{k=1}^{n} A_k(t) \cdot \frac{d^{n-k} u(t)}{dt^{n-k}} = f(t), \qquad (1)$$

$$\left. \frac{d^k u(t)}{dt^k} \right|_{t=0} = u^{(k)}(0), \quad k = 0,1,\ldots,n-1 \qquad (2)$$

where $u(t),\ f(t)-$ functions of scalar argument $t \in [0;T]$ with values in $D(A) = \cap D(A_k(t)) \subset H,\ \ A_k(t)-$ generally speaking, linear unbounded operators acting in $H$. The "$l-$ correctness" of problem (1), (2) is proved under certain assumptions on $A_k(t)$ operators.

## II. Main part

Suppose that in problem (1), (2) the $A_k(t)$ operators are defined on a constant everywhere dense set, are closed and strongly continuously differentiable up to $(n-k)$ order inclusive. Suppose that $Z(X_1, X_2)$ is the algebra of linear bounded operators acting from $X_1$ to $X_2$, $A^{(m)}{}_k(t)$ is the strong derivative of $A_k(t)$ operator of order $m$ and $G=\{U_h\}-$, and a strongly continuous group of unitary operators in $H$. By Stone's theorem, $G$ has a generating $A$ operator and $i^{-1}A-$ is a self-adjoint operator. We introduce a Hilbert space $H^l$ with the norm

$$\|u\|_l=\left\|\left(I-A^2\right)^{\frac{l}{2}}u\right\|_H, l>0.$$

Theorem: Let the operators $A_k^{(m)}(t)\in Z\left(H^l,H\right)$ and be $G$ group invariant, i.e.

$$U_hA_k^{(m)}(t)=A_k^{(m)}(t)\cdot U_h,\qquad h\in(-\infty;+\infty),\ 0\le m\le n-k.$$

Then the solution to Cauchy's problem (1), (2) satisfies the inequality

$$\|u\|\le\varepsilon\cdot\left\|S^{-1}u\right\|+\varphi(\varepsilon)\|Tu\|,\forall\varepsilon>0,$$

where

$$\varphi(\varepsilon)=M\sum_{n=0}^{\infty}\left(\frac{16M^2}{\varepsilon}\right)^{n+1}\left\|N^n\right\|,\quad T=S+N,$$

$$(Nu)(t)=\int_0^t\left[\sum_{m=0}^{n-1}\sum_{k=0}^{m}(-1)^{k+m}\frac{C_m^k}{(n-k-1)!}(t-\tau)^{n-k}A_{n-m}^{(m-k)}(\tau)S\right]u(\tau)d\tau,\quad S=\left(I-A^2\right)^{-\frac{l}{2}}$$

Evidence. The proof of the theorem is based on the idea of 3 theorem proof in [3]. To equation (1) we apply $n$ times the operator of inverse differentiation

$$u(t)+\frac{1}{(n-1)!}\left\{\int_0^t(t-\tau)^{n-1}A_1(\tau)u^{(n-1)}(\tau)d\tau+\ldots+\int_0^t(t-\tau)^{n-1}A_n(\tau)u(\tau)d\tau\right\}=$$

$$=\frac{1}{(n-1)!}\int_0^t(t-\tau)^{n-1}f(\tau)d\tau+\sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0).$$

In each integral of the last equation, the derivatives $u^{(k)}(t)$ we transfer to other factors. Then we get

$$u(t)+(N_1u)(t)=f_1(t),\qquad (3)$$

where

$$(N_1u)(t)=\int_0^t\left[\sum_{m=0}^{n-1}\sum_{k=0}^{m}(-1)^{k+m}\frac{C_m^k}{(n-k-1)!}(t-\tau)^{n-k-1}A_{n-m}^{(m-k)}(\tau)\right]u(\tau)d\tau,$$

$$f_1(t)=\frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau+\sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0)+$$

$$+\sum_{l=1}^{n-1}\sum_{k=1}^{n-l}\sum_{m=1}^{k}(-1)^m C_{k-1}^{m-1}\frac{t^{n+m-k-1}}{(n+m-k-1)!}A_l^{(m-1)}(0)\cdot u^{(n-l-k)}(0).$$

If we apply the operator $S$ to equation (3), then we obtain

$$Su(t)+(Nu)(t)=f_2(t), \qquad (4)$$

where $$f_2(t)=S\,f_1(t).$$

Now let's check that the operator $S$ satisfies the following conditions:

1) $S\in Z(H,H^l);\qquad SA_k^{(m)}(t)=A_k^{(m)}(t)S;$

2) $S$ operator as a scalar operator $H\to H$ (see [3] for the definition of a scalar operator);

3) $\ker S=0.$

Because

$$\|Su\|_l=\|u\|_H, \qquad (5)$$

then $S\in Z(H,H^l)$. Further, by virtue of the invariance of $A_k^{(m)}(t)$ with respect to $G$, $A_k^{(m)}(t)A=AA_k^{(m)}$ and, thus, the operators $A_k^{(m)}(t)$ commute with $S$.

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Therefore, condition 1 is satisfied. The $i^{-1}A$ operator is self-adjoint, therefore, the $S$ operator is Hermitian (all the more scalar). So condition 2 is also satisfied. Condition 3 follows from (5).
Moreover, since

$$S\in Z(H,H^l),\ A_k^{(m)}(t)\in Z(H^l,H),$$

then $SA_k^m(t)\in Z(H,H)$.

Thus, we have obtained equation (4), which satisfies all the conditions of Theorem 5 in [3]. Thus, we have proved the assertion of the theorem.
The theorem proved and Theorem 6 in [3] imply the following.

Consequence. If $(Su,u)\geq 0,\ \forall u\in H,$ then the estimate

$$\|u\| \le \varepsilon \left\| S^{-\delta} u \right\| + \varphi_\delta(\varepsilon) \|Tu\|, \quad \forall \varepsilon > 0, \forall \delta > 0$$

where

$$\varphi_\delta(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{-\frac{n+1}{\delta}} \left\| N^n \right\|.$$

Now, consider the Cauchy problem in $H = L_2[a,b]$ space for equation (1), when $A_k(t)-$ self-adjoint differential operators generated by differential expressions

$$l_k(u) = \sum_{l=0}^{m_k} a_{kl}(t) \frac{\partial^l u}{\partial x^l}$$

and boundary conditions

$$U_{k\nu} = 0, \quad \nu = 1, 2, \dots, m_0^k,$$

where $U_{k\nu}(u)-$ linear forms with respect to variables

$$u_a, u_a^{'}, \dots, u_a^{(m_k - 1)}, \quad u_b, u_b^{'}, \dots, u_b^{(m_k - 1)},$$

$a_{kl}(t)-$ scalar functions that are continuously differentiable for every $l$ $n-k$ times.

In this case, in order to obtain the type estimate of Theorem 1, it is necessary to prove the $S$ operator existence, which satisfies conditions 1–3 considered in the Theorem 1 proof. Let us consider the case when among the $A_k(t)$ operators there is an $A_\mu$ operator, which has the highest order and does not depend on $t$. In addition, we assume that $D(A_\mu) \subseteq D(A_k)$, где $D(A_k)$ - operator scope $A_k(t)$.

Theorem 2. Let the boundary conditions of the $A_\mu$ operator be the form

$$u_a = u_a^{'} = \dots = u_a^{\left(\frac{m_\mu}{2} - 1\right)} = u_b = u_b^{'} = \dots = u_b^{\left(\frac{m_\mu}{2} - 1\right)} = 0 \quad (6)$$

and $A_\mu u = 0$ has only one solution.

Then

$$Su = \int_a^b G(x;y) u(y) dy,$$

where $G(x;y)-$ Green's function operator $A_\mu$.

Evidence. By the hypothesis of the theorem, there is a unique Green's function [4] for which the following properties hold:

1. $G(x;y) = G(y;x)-$ symmetry.

2. $G(x;y)-$ is continuous and has continuous derivatives with respect to $x$ and $y$ up to the $(m_\mu - 2)-$ order inclusive for all values $x$ and $y$ from the interval $[a;b]$. 3. For any fixed $y$ $(x)$ from interval $[a;b]$ function $G(x;y)$ has continuous derivatives $(m_\mu - 1)-$ and $m_\mu -$ order in $x$ (in $y$ ) in each of the intervals $[a,y)$ and $(y,b]$( $[a,x)$ and $(x,b]$ ), the derivative $(m_\mu - 1)-$ of order has a jump at $x = y$

$$\frac{1}{a_{\mu m_\mu}}.$$

4. In each of the intervals $[a,y)$ and $(y,b]$( $[a,x)$ and $(x,b]$ ) function $G(x;y)$, considered as a function of $x$ (from $y$ ) satisfies the condition $l_\mu(G)=0$ and boundary conditions (6).

Property 2 implies the $S$ boundedness. $A_\mu$ does not depend on $t$, it follows that $G(x,y)$ also does not depend on $t$. Besides, $G(x,y)$ - symmetrical means $S$ - self-adjoint, therefore $S$ - scalar. That $\ker S = 0$ - obviously. Therefore, in order to prove the theorem, it remains to verify that $S$ commutes with the $A_k^{(m)}(t)$ operators. It is known [4] that a self-adjoint operator of even order. Since the coefficients of the operators $A_k^{(m)}(t)$ do not depend on $x$, must have, $A_k^{(m)}(t)$ have no derivatives of odd order. It's obvious that $S$ transfers $L_2[a,b]$ in $D(A_\mu)$, then it suffices to check that for any even $i \le m_\mu - 2$

$$\int_a^b G(x,y)u^{(i)}(y)dy = \int_a^b \frac{\partial^i G(x,y)}{\partial x^i}u(y)\,dy \qquad (7)$$

Bearing in mind property 2 the Green's function and integrating the left-hand side of equality (7) by parts, we obtain

$$\int_a^b G(x,y)u^{(i)}(y)dy = \left[ G(x,y)u^{(i-1)}(y) - \frac{\partial G(x,y)}{\partial y}\cdot u^{(i-2)}(y) + \right.$$

$$\left. +\ldots - \frac{\partial^{i-1}G(x,y)}{\partial y^{i-1}}u(y) \right]_a^b + \int_a^b \frac{\partial^i G(x,y)}{\partial y^i}u(y)\,dy.$$

## III. Conclusion

In the last equality, by virtue of (6) and the properties and Green's functions, the terms outside the integral are zero, and the integral, by virtue of property 1,2 is equal to the right integral in equality (7). So, $SA_k^{(m)}(t) = A_k^{(m)}(t)S$ . The theorem is proved.

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