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Fuzzy ϑ –BFGS Update for Numerical Optimization

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***Abstract* — In this paper, we develop a quasi-Newton method for unconstrained optimization problems with fuzzy functions. Also, we Propose the ϑ-BFGS method to fuzzy optimization problems. The generalized Hukuhara differentiability for fuzzy functions is employed. By using a Hessian approximation, we resolve the high computational cost of finding the Hessian in Newton method for fuzzy optimization problems. We will find the solutions of a fuzzy optimization problem. Finally, some numerical results support this claim and also indicate that the ϑ-BFGS update may be competitive with the BFGS update in general**

Keywords* — *Unconstrained fuzzy optimization problems; Quasi-Newton methods; Superlinear convergence; generalized Hukuhara differentiability,

I. INTRODUCTION

In optimizing real world systems, we deal with linear and nonlinear programming problems. In the conventional optimization problems, all of the coefficients are real numbers. However, in the real world, the coefficients involved in the objective and constraint functions are imprecise in nature and have to be stated in fuzzy sense to reflect the real world situation. Fuzzy optimization problems were first considered by Bellman and Zadeh [5]. Thereafter, Tanaka et al. [25] introduced the concept of fuzzy mathematical programming in a general level. Over the last decades, many researchers have studied optimization problems with fuzzy-valued objective functions. We refer to [1, 8, 9, 14, 15, 16, 17, 18, 19, 20] that have been done in this direction.

Pirzada and Pathak [22] proposed the Newton method for unconstrained optimization problems with fuzzy-valued functions. In their proposed method, they utilized Hukuhara differentiability of fuzzy-valued functions and max-ordering relation defined on the set of fuzzy numbers. Afterwards, Chalco-Cano et al. [6] addressed some of the difficulties of the method proposed by Pirzada and Pathak [22] and by using general- ized Hukuhara differentiability (gH-differentiability) of fuzzy functions they resolved these difficulties. More recently, Ghosh [10, 11, 12] proposed (quasi-)Newton methods for finding efficient solutions of optimization problems with interval-valued objective functions. The Newton method proposed in [6, 22] is well defined only when the optimization problem is convex. Moreover, for large scale optimization problems, computing the Hessian matrix is very hard and cost effective. Therefore, in this paper, we are going to provide a quasi-Newton method to resolve the computational cost in computing the Hessian of the Newton method [6, 22] for unconstrained fuzzy optimization problems. Resolving the computational cost is done through approximating the Hessian matrix by another matrix that is available at lower cost. Quasi-Newton methods are among the most widely used methods for solving nonlinear optimization problems, see [2, 21]. They are effective in solving a wide variety of small to mid- size optimization problems, in particular when the Hessian is hard to compute.

There are many different quasi-Newton methods, but they are all based on approximating the Hessian matrix by another matrix that is available at lower cost. Here, we extend the well-known BFGS method to fuzzy optimization problems. There are several advantages in this approach. At first, an approximation matrix can be found by using only first-derivatives information. Second, for problems that their Hessian is hard to compute, the proposed algorithm works efficiently. We show that the method converges superlinearly.

This paper is as follows. In Section 2, some Preliminaries and Basic properties and arithmetic of fuzzy sets are introduced. In Section 3, some results on gH-differentiability of fuzzy functions in Section 4 fuzzy optimization are provided. The proposed quasi-Newton method. To illustrate the efficiency of the proposed method, numerical examples are presented in Section 5. Finally, conclusions are given in Section 6.

II. Preliminaries And Basic Definitions

The concepts and notations of sets (crisp set), and then introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalization of the crisp sets. Convex set, $\alpha-$cut operation, cardinality of fuzzy number are also introduced.

Definition (2.1),[14] :-

A set $\widetilde{A}=\{(x,\mu_{\widetilde{A}}(x)), x\in X\}$ is called a fuzzy set where $\mu_{\widetilde{A}}(x)$ is the membership function of a fuzzy set A is defined by $\mu_{\widetilde{A}}(x): X\to[0,1]$,the value of $\mu_{\widetilde{A}}(x)$ is called the membership degree of x in X. The following example illustrates the membership function as a function of crisp set.

Definition (2.2),[5]:-

A fuzzy number is a fuzzy set which is a map $\tilde{u}: R\to[a,b]$, that satisfies

(1) u is upper semi-continuous function,

(2) u(x) = 0 outside some interval $[a,d]$

(3) There are a real numbers b,c such a $\le$ b $\le$ c $\le$ d that is

(i) u(x)is a monotonic increasing function on $[a,b]$.

(ii) u(x) is a monotonic decreasing function on $[c,d]$.

(iii) u(x) = 1 for all $x\in[b,c]$.

The set of all fuzzy numbers is denoted by E^1 and is a convex cone.

Definition (2.3),[5]:-

A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u},\overline{u})$of function $\underline{u}(\alpha),\overline{u}(\alpha)$, $0\le\alpha\le1$, which satisfies the following conditions:

(i) . $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over [0, 1].

(ii). $\overline{u}(\alpha)$ is a bounded left continuous non- increasing function over [0, 1].

(ii) . $\underline{u}(\alpha)\le\overline{u}(\alpha)$,$0\le\alpha\le1$.

Definition(2.4),[3]:-

The arbitrary fuzzy number $\tilde{u}=(\underline{u}(\alpha),\overline{u}(\alpha))$, $\tilde{v}=(\underline{v}(\alpha),\overline{v}(\alpha))$, $0\le\alpha\le1$ and scalar k, the operations defended as follows:

(i) . $\underline{(u+v)}(\alpha)$= $(\underline{u}(\alpha)+\underline{v}(\alpha))$,$\overline{(u+v)}(\alpha)$ =$\overline{u}(\alpha)+\overline{v}(\alpha))$,

(ii). $\underline{(u-v)}(\alpha)$= $(\underline{u}(\alpha)$-$\underline{v}(\alpha)$,$\overline{(u-v)}(\alpha)$ = $(\overline{u}(\alpha)$-$\overline{v}(\alpha))$,

$$\text{(iii)}\quad .\ k\breve{u}=\begin{cases} Min\left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k\ge0\\ Max\left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k<0\end{cases}\qquad(1.8)$$

Definition (2.5),[8]:-

The family of all fuzzy number E^n, denotes all nonempty compact convex fuzzy subset of R^n. Let $I=[a,b]$ be a compact interval:

$$E^n=\{p: R^n\to I\}$$

such that p satisfies the following

i) p is normal

ii) p is fuzzy convex,

iii) p is upper semi continuous, i.e, the $\alpha-$level sets $[p]_\alpha$are closed for each $\alpha\in[0,1]$,

iv) $[p]^0$=cl$\{x\in R^n|p(x)>0\}$ is compact.

Where the $\alpha-$level sets $[p]^\alpha$ is defined by: $[p]^\alpha=\{x\in R^n|p(x)\ge\alpha\}$ for $0<\alpha\le1$ and $[p]^0$ for $\alpha=0$.

Then from (i)-(iv), it follows that $[p]^\alpha\in E^n$ for all $0\le\alpha\le1$.

Definition (2.6),[9]:-

Let R be the set of real number and $\widetilde{P}(R)$ all fuzzy subsets defined on R , defined the fuzzy number $\tilde{a}\in E^1$ as follows :

1- $\tilde{a}$ is normal , that is there exists $x\in R$ such that $\mu_{\tilde{a}}(x)=1$

2- For every $\alpha\in(0,1]$, $a_\alpha=\{x:\mu_{\tilde{a}}(x)\ge\alpha\}$ is closed interval , denoted by:

$$[a^-{}_\alpha, a^+{}_\alpha]$$

Using Zaheh's notation $\tilde{a}\in F(R)$ is the fuzzy set on R defined by:

$$\tilde{a}=\cup_{\alpha\in[0,1]}a_\alpha=\cup_{\alpha\in[0,1]}\alpha[a^-{}_\alpha,a^+{}_\alpha]$$

Definition(2.7), [6]: -

Let $\tilde{a},\tilde{b}$ and $\tilde{c}\in E^1$ we define the following operation as:

1. $\tilde{c}=\tilde{a}+\tilde{b}$ if $c^-{}_\alpha=a^-{}_\alpha+b^-{}_\alpha$ and $c^+{}_\alpha=a^+{}_\alpha+b^+{}_\alpha$, for every $\alpha\in[0,1]$

2. $\tilde{c} = \tilde{a} - \tilde{b}$ if $c^-_\alpha = a^-_\alpha - b^+_\alpha$ and $c^+_\alpha = a^+_\alpha - b^-_\alpha$, for every $\alpha \in [0,1]$

3. $\tilde{c} = \tilde{a}.\tilde{b}$ if
$c^-_\alpha = \mathrm{Min}[a^-_\alpha.b^-_\alpha, a^-_\alpha.b^+_\alpha, a^+_\alpha, b^-_\alpha, a^+_\alpha.b^+_\alpha]$

$c^+_\alpha = \mathrm{Mxa}[a^-_\alpha.b^-_\alpha, a^-_\alpha.b^+_\alpha, a^+_\alpha, b^-_\alpha, a^+_\alpha.b^+_\alpha]$, for every $\alpha \in (0,1]$

4. $\tilde{c} = \tilde{a}/\tilde{b}$ if $c^-_\alpha = \mathrm{Min}[a^-_\alpha / b^-_\alpha, a^-_\alpha / b^+_\alpha, a^+_\alpha / b^-_\alpha, a^+_\alpha / b^+_\alpha]$
$c^+_\alpha = \mathrm{Mxa}[a^-_\alpha / b^-_\alpha, a^-_\alpha / b^+_\alpha, a^+_\alpha / b^-_\alpha, a^+_\alpha / b^+_\alpha$
for every $\alpha \in (0,1]$, excluding the case $b^-_\alpha = 0$ or$b^+_\alpha = 0$

For every $k \in R$ and $\tilde{a} \in E^1$, $k\tilde{a} = \cup_{\alpha\in[0,1]} \alpha[ka^-_\alpha, ka^+_\alpha]$ if $k \geq 0$

$k\tilde{a} = \cup_{\alpha\in[0,1]} \alpha[ka^+_\alpha, ka^-_\alpha]$ if $k < 0$

5. $\tilde{a} \leq \tilde{b}$ if $a^-_\alpha \leq b^-_\alpha$ and $a^+_\alpha \leq b^+_\alpha$ for every $\alpha \in (0,1]$
6. $\tilde{a} \leq \tilde{b}$ if $a^-_\alpha \leq b^-_\alpha$ and there exists $\alpha \in (0,1]$ such that $a^-_\alpha < b^-_\alpha$ or $a^+_\alpha < b^+_\alpha$
7. 8. $a^+_\alpha = b^+_\alpha$ if $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$ for every $\alpha \in (0,1]$

Definition (2.8)[7]:- Let u and v be two fuzzy number in E^n. $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$ are two intervals in R, for all $\alpha \in [0,1]$. We define

$$u \leq v \Leftrightarrow [u]^\alpha \leq [v]^\alpha, \forall\alpha \in [0,1] \leftrightarrow \underline{u}^\alpha \leq \underline{v}^\alpha and\ \overline{u}^\alpha \leq \overline{v}^\alpha, \forall\alpha \in [0,1]$$

And

$$u < v \leftrightarrow u \leq v\ and\ u \neq v \leftrightarrow [u]^\alpha \leq [v]^\alpha, \forall\alpha \in [0,1], and\ \exists\alpha^* \in [0,1] s.t\ \underline{u}^{\alpha^*} < \underline{v}^{\alpha^*}\ and\ \overline{u}^{\alpha^*} < \overline{v}^{\alpha^*}$$

Definition (2.9) [24] Given $u, v \in E^n$. The fuzzy number w is called the generalized Hukuhara difference (gH-difference for short) between u and v, if

$$u \ominus_{gh} v = w \leftrightarrow \begin{cases} (i) u = v + w \\ (ii) v = u - w \end{cases}$$

Using the α level we will get

$[u \ominus_{gh} v]^\alpha = [u]^\alpha \ominus_{gh} [v]^\alpha = [\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}], \forall\alpha \in [0,1]$ where $[u]^\alpha \ominus_{gh} [v]^\alpha$ is the gH-difference between two intervals

III. Differentiable Fuzzy Functions[4,6,10,11]

Let X denotes an open subset of R^n. A mapping $F : X \to E^n$ is said to be a fuzzy function defined on X.

Let X_c denote the family of all bounded closed intervals in R. We associate with each fuzzy function $F : X \to E^n$ the family of interval-valued functions $F_\alpha : X \to X_c$ given by $F_\alpha(x) = [F(x)]^\alpha$. For each $\alpha \in [0, 1]$, we denote

$$F_\alpha(x) = [F(x)]^\alpha = [\underline{f}^\alpha, \overline{f}^\alpha].$$

The endpoint function $\underline{f}^\alpha, \overline{f}^\alpha : X \to R$ are said to be upper and lower function of $F_\alpha(x)$, respectively.

Definition (3.1) [25] Let $X \subset R$ and $F : X \to E^n$ be a fuzzy function. Also, assume that $x_0 \in X$ and h be such that $x_0 + h \in X$. The generalized Hukuhara derivative (gH-derivative) of F at x_0 is defined as

$$F'(x_0) = \lim_{h\to 0} \frac{F(x_0 + h) \ominus_{gh} F(x_0)}{h} \quad (3.1)$$

If $F'(x_0) \in E^n$ satisfying (3.1) exists, then F is said to be generalized Hukuhara differentiable (gH-differentiable) at x_0. If F is gH-differentiable at any $x \in X$, we say that F is gH-differentiable over X.

Definition (3.2) [25] Let X be an open set in R. An interval-valued function $F : X \to X_c$ is gH-differentiable at $x_0 \in X$, if (3.1) exists with respect to the limit in the metric space (X_c, H), where the difference is given by the gH-difference between intervals.

Theorem (3.1) [2] *Let $F : X \to E^n$ be a fuzzy function. If F is gH-differentiable, then the interval-valued function $F_\alpha : X \to X_c$ is gH-differentiable for each $\alpha \in [0, 1]$. Moreover*

$$F'_\alpha(x) = [F'(x)]^\alpha = [(\underline{f}^\alpha)'(x), (\overline{f}^\alpha)'(x)]$$

Theorem (3.2) [6] *Let $F : X \longrightarrow E^n$ be a fuzzy function. If F is gH-differentiable at $x_0 \in X$ then, for each $\alpha \in [0, 1]$, one of the following cases holds*

(i). $\underline{f}^\alpha and\ \overline{f}^\alpha\ are\ differentiable\ at\ x_0\ and$

$$[F'(x_0)]^\alpha = [\min\{(\underline{f}^\alpha)'(x_0), (\overline{f}^\alpha)'(x_0)\}, \max\{(\underline{f}^\alpha)'(x_0), (\overline{f}^\alpha)'(x_0)\}];$$

(ii). $(\underline{f}^\alpha)'_-(x_0), (\overline{f}^\alpha)'_-(x_0), (\underline{f}^\alpha)'_+(x_0)\ and\ (\overline{f}^\alpha)'_+(x_0)$

exist and satisfy

$$\left(\underline{f}^{\alpha}\right)'_{-}(x_0) = \left(\overline{f}^{\alpha}\right)'_{+}(x_0) \; and \; \left(\underline{f}^{\alpha}\right)'_{+} = (x_0)\left(\overline{f}^{\alpha}\right)'_{-}(x_0)$$

.

$$[F'(x_0)]^{\alpha}$$

$$= [\min\left\{\left(\underline{f}^{\alpha}\right)'_{-}(x_0), \left(\overline{f}^{\alpha}\right)'_{-}(x_0)\right\}, \max\left\{\left(\underline{f}^{\alpha}\right)'_{-}(x_0), \left(\overline{f}^{\alpha}\right)'_{-}(x_0)\right\}]$$

$$=[\min\{\left(\underline{f}^{\alpha}\right)'_{+}(x_0), \left(\overline{f}^{\alpha}\right)'_{+}(x_0)\}, \max\{\left(\underline{f}^{\alpha}\right)'_{+}(x_0), \left(\overline{f}^{\alpha}\right)'_{+}(x_0)\}]$$

Definition (3.2)[25]:- Let $F: X \to E^n$ be a fuzzy function and let $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ be a fixed element of X. Consider the fuzzy function $h(x_i) = F(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_n^0)$. If h is gh-differentiable at x_i^0 then we say that F has the ith partial gh-derivetive at x^0 (denoted by $\frac{\partial F}{\partial x_i}(x^0) = \big(h'\big)(x^0)\big)$.

Definition(3.3)[20]:- Let F be a fuzzy function defined on $R^n \supseteq X$ and $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X$ be a fixed element of X. F is said to be gh-differentiable at x^0 if all partial gh-derivative $\frac{\partial F}{\partial x_1}(x^0), \frac{\partial F}{\partial x_2}(x^0), \dots, \frac{\partial F}{\partial x_n}(x^0)$ exist in some neighborhood of x^0 and are continuous at x^0

Definition(3.4)[25]:- Let $F: X \to E^n$ be a fuzzy function. The gradient of F at x^0, denoted by $\widetilde{\nabla} F(x^0)$ is defined by

$$\widetilde{\nabla} F(x^0) = \left(\left(\frac{\partial F}{\partial x_1}\right)(x^0), \left(\frac{\partial F}{\partial x_2}\right)(x^0), \dots, \left(\frac{\partial F}{\partial x_n}\right)(x^0)\right).$$

The $\alpha - Level$ set of $\widetilde{\nabla} F(x^0)$ is defined and denoted by

$$[\nabla F(x^0)]^{\alpha} = \left(\left[\frac{\partial F}{\partial x_1}\right]^{\alpha}(x^0), \left[\frac{\partial F}{\partial x_2}\right]^{\alpha}(x^0), \dots, \left[\frac{\partial F}{\partial x_n}\right]^{\alpha}(x^0)\right)$$

When

$$\left[\frac{\partial F}{\partial x_i}\right]^{\alpha} = [\frac{\partial \underline{F}^{\alpha}}{\partial x_i}, \frac{\partial \overline{F}^{\alpha}}{\partial x_i}]$$

Theorem (3.3)[25]:- Let $F: X \to E^n$ be a fuzzy function. If F is gh-differentiable at $x^0 \in X$ then, for each $\alpha - level$, the real-valued function $\underline{f}^{\alpha} + \overline{f}^{\alpha}: X \to R$is differentiable at x^0.

$$\frac{\partial \underline{F}^{\alpha}}{\partial x_i}(x^0) + \frac{\partial \overline{F}^{\alpha}}{\partial x_i}(x^0) = \frac{\partial\left(\underline{f}^{\alpha} + \overline{f}^{\alpha}\right)}{\partial x_i}(x^0)$$

Definition (3.5)[15]:- Let $F: X \to E^n$ be a fuzzy function. If gradient of F, $\widetilde{\nabla} F$, is itself gh-differentiable at x^0, that is, for each I, the function $\frac{\partial F}{\partial x_i}: X \to E^n$ is gh- differentiable at x^0, then we say that F is twice gh-differentiable at x^0, Denoted the gh-partial derivative of $\frac{\partial F}{\partial x_i}$ by

$$\widetilde{D}_{ij}^2 F(x^0) = \frac{\partial^2 F}{\partial x_i x_j}(x^0), \qquad i \neq j$$

And

$$\widetilde{D}_{ii}^2 F(x^0) = \frac{\partial^2 F}{\partial {x^2}_i}(x^0), \qquad i = j$$

If for each i,j=1,2,…,n, the crose-partial derivatives $\frac{\partial^2 F}{\partial x_i x_j}$ is continuous from X to E^n, we say that F is twice continuously differentiable.

Theorem Let $F: X \to E^n$ be a fuzzy function. If F is m-times differentiable at $x^0 \in X$ then , for each $\alpha \in [0,1]$, the real- valued function $\underline{f}^{\alpha} + \overline{f}^{\alpha}: X \to R$ is m-times differentiable at x^0.

IV. FUZZY OPTIMIZATION[12,13, 16,17]

We consider the following fuzzy optimization problem (FOP):

(FOP)$min_{x \in X} F(x)$

Where the objective function $F: X \to E^n$is a fuzzy- valued function and $X \in E^n$ is the domain of F which is assumed to be an open set. In the remainder of the paper we assume that F is gh-differentiable.

Definition(4.1)[16] Let $X \subset E^n$ be open set. We say that $x^* \in X$ is locally non dominated solution of FOP if there exists no $x \in N_{\epsilon}(x^*) \cap X$ such that $F(x) < F(x^*)$, where $N_{\epsilon}(x^*)$ is an $\epsilon - nighborhood\ of\ x^*$.

Theorem(4.1)[17] Let $X \subset E^n$ be an open set and $F: X \to E^n$ be a fuzzy function. If x^* is local minimizer of the real-valued function $\underline{f}^{\alpha} + \overline{f}^{\alpha}$ for all $\alpha \in [0,1]$, thenx^* is locally nondominated solution of the FOP

Theorem(4.2)[21]:- Let $\alpha \in [0,1]$ and the real-valued function $\underline{f}^{\alpha} + \overline{f}^{\alpha}: X \to E^n$ be differentiable at x^0. If there is a vector d such that $(\nabla\left(\underline{f}^{\alpha} + \overline{f}^{\alpha}\right)(x^0))^T, d < 0$, then there exists a $\delta > 0$ such that $(\underline{f}^{\alpha} + \overline{f}^{\alpha})(x^0 + kd) < (\underline{f}^{\alpha} + \overline{f}^{\alpha})(x^0)$ for each $k \in (0, \gamma)$. So, that dis a descent direction of $\underline{f}^{\alpha} + \overline{f}^{\alpha}$.

Modified BFGS update for solving fuzzy optimization problem.

By using $\alpha - level$ to find the upper and lower respectively.

Remark (4.1)[18,19,20]:-

Relation $y^{\alpha}(x^{i})^{T}s_{i} > 0$ is the fuzzy curvature condition. The fuzzy curvature fuzzy condition for all $\alpha \in [0,1]$ implies that

$$\int_{0}^{1} y^{\alpha}(x^{i})^{T}s_{i}d\alpha = y(x^{i})^{T}s_{i} > 0$$

The curvature condition. Since the Newton Method, the Hessian matrix $\nabla^{2}(\overline{f}^{\alpha} + \underline{f}^{\alpha})$ for all $\alpha \in [0,1]$ is symmetric positive definite matrix , the approximation matrix $(\overline{B}^{\alpha} + \underline{B}^{\alpha})(x^{i})$ be positive definite matrix $\forall\alpha \in [0,1]$. This is possible if the fuzzy curvature condition satisfies, than $(\overline{B}^{\alpha} + \underline{B}^{\alpha})(x^{i})$ is positive definite. The monition relations, we will modifies $\gamma, \vartheta - BFGS$ update method for solving our problem formulation for the FOP

$$(\overline{B}^{\alpha i} + \underline{B}^{\alpha i}) = \left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right) + \frac{y^{\alpha}y^{\alpha T} - \left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)}{(y^{\alpha} + \left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1})^{T}}$$

i=0,1,…,

Where , Hessian matrix $\nabla^{2}(\overline{f}^{\alpha} + \underline{f}^{\alpha})$ represent the second derivative of the objective fuzzy function $\tilde{f}$

$\nabla(\overline{f}^{\alpha} + \underline{f}^{\alpha})$ represent the first derivative of the objective fuzzy function $\tilde{f}$

$(\overline{B}^{i} + \underline{B}^{i})$ represent the next approximation of fuzzy Hessian matrix

$\overline{(B^{\alpha(i-1)}} + (\underline{B^{\alpha(i-1)}})$ represent the current Hessian matrix

$$y^{i}(x^{(i)}) = \left(\nabla\left(\overline{f}^{\alpha} + \underline{f}^{\alpha}\right)x^{(i)} - \nabla\left(\overline{f}^{\alpha} + \underline{f}^{\alpha}\right)x^{(i-1)}\right)$$

$$s_{i} = x^{(i)} - x^{(i-1)}$$

$x^{(i-1)}, x^{i} \in R^{n}$ represent the lower and upper current approximation to the original solution

$x^{(i)}, x^{(i-1)} \in R^{n}$ represent the next lower and upper approximation to the original solution

For fuzzy example, if $x \in R^{2}$ *then the lower and upper matrix*=

$$\nabla(\overline{f} + \underline{f}) = \begin{bmatrix} \frac{\partial(\overline{f}+\underline{f})}{\partial x_{1}} \\ \frac{\partial(\overline{f}+\underline{f})}{\partial x_{2}} \end{bmatrix},$$

And the lower and upper Hessian matrix =

$$\nabla^{2}(\overline{f} + \underline{f}) = \begin{bmatrix} \frac{\partial^{2}(\overline{f} + \underline{f})}{\partial x_{1}^{2}} & \frac{\partial^{2}(\overline{f} + \underline{f})}{\partial x_{1}\partial x_{2}} \\ \frac{\partial^{2}(\overline{f} + \underline{f})}{\partial x_{2}\partial x_{1}} & \frac{\partial^{2}(\overline{f} + \underline{f})}{\partial x_{2}^{2}} \end{bmatrix},$$

By performing a simple summarization procedure from a rank one update, this method is locally convergent and *possesses* a local superlinearly convergent rate. We can write upper and lower equations, as follows.

$$(\overline{B^{\alpha}} + \underline{B^{\alpha}}^{i}) = \left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right) + \frac{y^{\alpha T}s_{i-1}}{y^{\alpha T}s_{i-1}}\frac{y^{\alpha}y^{\alpha T}}{y^{\alpha T}s_{i-1} + s_{i-1}{}^{T}\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}} - \frac{s_{i-1}{}^{T}\left(\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)\right)s_{i-1}}{s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}}\frac{\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)}{y^{\alpha T}s_{i-1} + s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}}$$

i=0,1,…,

$$(\gamma, \vartheta) = \left(\frac{y^{\alpha T}s_{i-1}}{y^{\alpha T}s_{i-1}}, \frac{s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}}{s_{i-1}{}^{T}\left(\overline{B}^{\alpha(i-1)} + \underline{B}^{\alpha(i-1)}\right)s_{i-1}}\right) \quad \alpha \in [0,1], i = 0,1,2, \ldots$$

By using Remark 4.1, we will have

We rewrite equation (2) we get

$$(\overline{B} + \underline{B}^{i}) = \left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right) + \gamma\frac{yy^{T}}{y^{T}s_{i-1} + s_{i-1}{}^{T}\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}} - \vartheta\frac{\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}s_{i-1}{}^{T}\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)}{y^{T}s_{i-1} + s_{i-1}{}^{T}\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}}$$

The modified BFGS update. Is symmetric but does not satisfy the quasi- newton condition.

We take the upper and lower parameter γ *and* ϑ as follows

2. $\vartheta - BFGS$ update for solving fuzzy optimization problems

We consider the following unconstrained fuzzy optimization problems

(FOP)$Min_{x\in R^{n}}F(x)$ we assume that.

Assumption (4.1)[24]:-

1- $(\underline{f}^{\alpha} + \overline{f}^{\alpha})(x)$ is twice continuously differentiable

2- $(\underline{f}^{\alpha} + \overline{f}^{\alpha})(x)$ is uniformly convex, there are fuzzy positive constant m *and* M such that .

$m\|z\|^{2} \leq z^{T}G(x)z \leq M\|z\|^{2}$

For all $x, z \in R^{n}$ where G is denoted the Hessian matrix of($\underline{f}^{\alpha} + \overline{f}^{\alpha}$) and $\|.\|$ denotes the Euclidean norm . Not that the assumption implies that ($\underline{f}^{\alpha}, \overline{f}^{\alpha}$) has a unique minimize x^{i}, for upper and unique minimize x^{i}, for lower

The . $\gamma,\vartheta - BFGS$ update consist of iteration of the form $x^i = x^{(i-1)} + k^{(i-1)}$,i=0,1,…

Where $p^{(i-1)}$ is a search direction of the form

$$p^{(i-1)} = -B^{-1}\nabla(\overline{f}^{\alpha} + \underline{f^{\alpha}}), \quad , \text{i=1,2,3,4,}$$

The $\nabla(\overline{f}^{\alpha} + \underline{f^{\alpha}})$ is the gradient of($\overline{f}^{\alpha} + \underline{f}^{\alpha}$) at x and the Hessian approximation($\underline{B} + \overline{B}$) is update by the $\gamma,\vartheta - FBGS$ formula

$$(\overline{B}^i + \underline{B}^i) = \vartheta\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right) + \gamma\frac{yy^T}{y^T s_{i-1}} - \vartheta\frac{\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}s_{i-1}^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)}{s_{i-1}^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}}$$

i=0,1,2,…

Where

$$y^i = \nabla(\overline{f}^{\alpha} + \underline{f^{\alpha}})(x)^{(i)} - \nabla(\overline{f}^{\alpha}(\overline{x})^{(i-1)} + \underline{f^{\alpha}}(x)^{(i-1)})$$

$s^i = (x)^{(i)} - (x)^{(i-1)}$ and $(\underline{B} + \overline{B})$ is symmetric and positive definite γ *and* ϑ will be given later. The scalar k , are positive step length chosen by a line search so that the iteration and wolfe condition are satisfies .

$$(\overline{f}^{\alpha} + \underline{f^{\alpha}})(\underline{x}^{(i)} + \underline{kp}^{(i)}) \le (\overline{f}^{\alpha} + \underline{f}^{\alpha})(x)^{(i)} + \rho_1 k(\nabla(\overline{f}^{\alpha} + \underline{f^{\alpha}})(\overline{x})^{(i)})^T p^{(i)},$$

$$\nabla(\overline{f}^{\alpha} + \underline{f}^{\alpha})(x^{(i)} + kp^{(i)}))^T p^{(i)} \ge \rho_2(\nabla\left(\overline{f}^{\alpha} + \underline{f^{\alpha}}\right))^T \underline{p}^{(i)},$$

Where $0 < \rho_1 < \rho_2 < 1$ *and* $\rho_1 < \frac{1}{2}, \dots$

We now show that the $\gamma,\vartheta - BFGS$ update preserve the positive definiteness if $\gamma > 0$ *and* $\vartheta > 0$

<u>Lemma (4.1)[24]:-</u> if $(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})$are positive definite then$(\overline{B}^{(i)} + \underline{B}^{(i)})$, defined by (8), also positive definite provide that $\vartheta > 0$ *and* $\gamma = 1$

Proof:-

Since $\underline{B}^i, \overline{B}^i$ of BFGS update is positive definite then by theorem 3 Borodlie , $\underline{B}^i, \overline{B}^i$ of BFGS update can be write as follows.

$$(\overline{B}^i + \underline{B}^i)_{BFGS} = [I + u\,v^T](\overline{B}^{(i-1)} + \underline{B}^{(i-1)})[I + uv]^T, \text{where } u, v \in R^n.$$

So the $(\overline{B}^i + \underline{B}^i)$ of the $\gamma,\vartheta - BFGS$ can be write as follows

$$(\overline{B}^i + \underline{B}^i)_{\gamma,\vartheta-BFGS} = [I + u\,v^T]\,\vartheta(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})[I + uv]^T, \text{where } u, v \in R^n.$$

Where i=1,2,…

Given $x \in R^n$ *and* $x \neq 0$

$$x^T(\overline{B}^i + \underline{B}^i)_{\gamma,\vartheta-BFGS}\,x = x^T[I + u\,v^T]\vartheta(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})[I + uv]^T x$$

$$= \vartheta[(I + u\,v^T)^T x]^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})[(I + uv^T)^T x] > 0$$

Since the BFGS update has q-superlinear convergence property, so $\underline{\vartheta},\overline{\vartheta} - BFGS$ update eq(8) with lower and upper parameter given by .

$$\vartheta = \left[\frac{s_{i-1}^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s_{i-1}}{y^T s_{i-1}}\right]^{\frac{1}{n-1}}, \underline{\gamma} = 1, n = 2$$

Also has q-superlinear convergence for upper and lower side .The following result can be found in Byrd.

<u>Lemma (4.2)[16,24]:-</u> Under assumption 1, we have

$$m\|s\|^2 \le y^T s \le M\|s\|^2$$

$$\frac{\|y\|^2}{y^T s} \le M$$

Following Byrd and Nocedal , we define the angel between s *and* $\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s$, θ *by*

$$\cos\theta = \frac{s^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s}{\|s\|\left\|\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s\right\|}$$ And the Rayleigh quotient

$$q = \frac{s^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s}{s^T s}$$

Using lemma (4.2) :- we estimate the trance of upper and lower Hessian approximation

$$tr\left((\overline{B}^i + \underline{B}^i)_{\gamma,\vartheta-BFGS}\right) = tr\left(\vartheta(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})\right) - \vartheta\frac{\left\|(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})\underline{s}\right\|^2}{s^T\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right)s} + \frac{\|y\|^2}{s^T y}$$

$$\le tr\left(\vartheta(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})\right) - \frac{\vartheta q}{\cos^2\theta} + \frac{\|y\|^2}{s^T y}$$

Now we will show that the $\gamma,\vartheta - BFGS$ update has a strong property with respect to the determinate

<u>Lemma (4.3) [24]</u> :-For the $\gamma,\vartheta - BFGS$ update with γ *and* ϑ using eq (11), if $(\overline{B}^i + \underline{B}^i)$ are positive definite the det($(\overline{B}^i + \underline{B}^i)_{\gamma,\vartheta-BFGS}$)=det($(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})$)

Proof:-

Using the rank-one update formula twice, we get

We proof the lower side, we have

$$\det((\overline{B}^i + \underline{B}^i)_{\gamma,\vartheta-BFGS}) = \det\left(\vartheta(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})\right)[(1-\gamma) + (1-\gamma)\frac{y^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})^{-1}y}{\vartheta s^T y} + \gamma\frac{y^T s}{\vartheta s^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})s}$$

$$= \vartheta^{n-1}\det(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}) \frac{y^T s}{s^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})s}$$

$$= [\left(\frac{s^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})s}{y^T s}\right)^{\frac{1}{n-1}}]^{n-1}\det((\overline{B}^{(i-1)} + \underline{B}^{(i-1)})) \frac{y^T s}{s^T(\overline{B}^{(i-1)} + \underline{B}^{(i-1)})s}$$

$$= \det\left(\overline{B}^{(i-1)} + \underline{B}^{(i-1)}\right), i = 0,1,2,\ldots$$

V. NUMERICAL EXAMPLES

In this paper we will give some illustrative examples . we use the stopping condition $\varepsilon > 0$ and the initial condition $x^i = \begin{bmatrix}0\\3\end{bmatrix}$, $and\ B^i = I, i = 0,1,2,\ldots$

Example 1

Consider the nonlinear programming problem with fuzzy function

$$(FOP) min_{x\in R^2}\ \tilde{F}(x)$$

Where

$\tilde{F}(x) = (\underline{f}^\alpha(x) + \overline{f}^\alpha(x))$

Now we will use definition for $\alpha - cute$ ( ) we will get the upper and lower function

$(\overline{f}^\alpha(x) + \underline{f}^\alpha)(x) = x_1^4 + 25x_1^2 - 19.8x_1x_2 + 4x_2^2 - 32x_1 - 32\alpha$

Using Remark (4.1), we get

$\int_0^1 (\overline{f}^\alpha(x) + \underline{f}^\alpha)(x)d\alpha = x_1^4 + 25x_1^2 - 19.8x_1x_2 + 4x_2^2 - 32x_1 + 16$

$$\nabla\left(\overline{f} + \underline{f}\right)(x) = \begin{pmatrix}4x_1^3 + 50x_1 - 19.8x_2 - 32x_1\\ -19.8x_1 + 8x_2\end{pmatrix}$$

In maple (version 18) we written programing and summary for fuzzy problem of the results is provided . Considering initial matrix point x = [0,3] and the stopping condition | |less than 0.05. the solution for this fuzzy problem we use the numerical develop ϑ - BFGS and we can reach to the absolute error with 4 step. In order to reduce the risk that unrepresentative results might be obtained on a particular problem by the choice of a particularly fortunate or unfortunate starting point, each problem was solved for a variety of starting point. Convergence was assumed when a point was located where in all cases. The entries in each column for the various methods denote the number of iterations required achieving convergence, followed by the number of function evaluation and the value of the minimum. The symbol F means that the update filed to get the minimum of the function. All the starting points were chosen to fuzzy comply with the requirement that the BFGS method, as implemented in our program, should converge from this point. Finally, an asterisk following the initial point indicates that method converged to a local minimum from that initial point. our develop $\vartheta-$ BFGS is so fast to satisfies the converge and reach less than and equal to the stopping condition . In table 1 we will numerical comparison between normal BFGS and our develop BFGS.

TABLE 1 CONVERGENCE WAS ASSUMED WHEN A POINT WAS LOCATED AND COMPARISON BETWEEN THE BFGS AND $\vartheta - BFGS$

BFGS					$\gamma,\vartheta - BFGS$				
i	x^i	$(\overline{f}^\alpha + \underline{f}^\alpha)$	$\nabla(\overline{f}^\alpha + \underline{f}^\alpha)$	p^i	x^i	$(\overline{f}^\alpha + \underline{f}^\alpha)$	p^i	ϑ^i	$\left\|\left(\nabla f(x^i)\right)^T \nabla f(x^{i-1})\right\| < 0.000005$
0	$\begin{bmatrix}0\\3\end{bmatrix}$	52.000	$\begin{bmatrix}-91.4\\24.00\end{bmatrix}$	$\begin{bmatrix}91.4\\-24.0\end{bmatrix}$	$\begin{bmatrix}0\\3\end{bmatrix}$	52.000000	$\begin{bmatrix}91.4\\-24.0\end{bmatrix}$	0.0156	8929.96000000000
1	$\begin{bmatrix}1.4281\\2.6250\end{bmatrix}$	-21.216	$\begin{bmatrix}-0.9179\\-7.2769\end{bmatrix}$	$\begin{bmatrix}2.3441\\6.8267\end{bmatrix}$	$\begin{bmatrix}1.41647\\2.62806\end{bmatrix}$	-21.221823	$\begin{bmatrix}153.78\\443.94\end{bmatrix}$	0.0038	241.995338240162
2	$\begin{bmatrix}2.0142\\4.3317\end{bmatrix}$	-28.269	$\begin{bmatrix}15.6247\\-5.2268\end{bmatrix}$	$\begin{bmatrix}-0.0804\\0.5337\end{bmatrix}$	$\begin{bmatrix}2.0022\\4.3189\end{bmatrix}$	-28.385164	$\begin{bmatrix}14.699\\-5.091\end{bmatrix}$	0.1872	52.7041335022081
3	$\begin{bmatrix}1.9337\\4.8654\end{bmatrix}$	-30.011	$\begin{bmatrix}-2.7243\\0.6352\end{bmatrix}$	$\begin{bmatrix}0.0281\\-0.0073\end{bmatrix}$	$\begin{bmatrix}1.8515\\4.3711\end{bmatrix}$	-29.612371	$\begin{bmatrix}0.2955\\1.3283\end{bmatrix}$	0.3589	3.200853582377837
4	$\begin{bmatrix}1.9618\\4.8581\end{bmatrix}$	-30.050	$\begin{bmatrix}0.1029\\0.0207\end{bmatrix}$	$\begin{bmatrix}-0.0038\\-0.0130\end{bmatrix}$	$\begin{bmatrix}1.9576\\4.8478\end{bmatrix}$	-30.050501	$\begin{bmatrix}0.00284\\-0.0008\end{bmatrix}$	$3.536x10^{10}$	0.0190487883654
5	$\begin{bmatrix}1.9580\\4.8451\end{bmatrix}$	-30.051	$\begin{bmatrix}-0.0043\\-0.0082\end{bmatrix}$	$\begin{bmatrix}0.0005\\0.0023\end{bmatrix}$	$\begin{bmatrix}1.9586\\4.8475\end{bmatrix}$	-31.0760	$\begin{bmatrix}0.00284\\-0.0008\end{bmatrix}$	$5.879x10^{17}$	0.00000317727862
6	$\begin{bmatrix}1.9585\\4.8474\end{bmatrix}$	-30.051	$\begin{bmatrix}10^{-3}x0.3526\\10^{-3}x0.0912\end{bmatrix}$	$\begin{bmatrix}10^{-5}\ x\ 0.2776\\-10^{-5}\ x\ 0.3342\end{bmatrix}$					

The quasi-newton method proposed in [22,23] could be applied for this numerical example. We employing BFGS, with same initial condition criterion is in a slower after 7 iteration, with our developed the $\vartheta - BFGS$ method is faster converge after 5 iteration. As it can seen in this example and also the value of and B will updating always before find new value and iterations , for numerical fuzzy optimization problem that explicitly computing of the update the matrix B and the invers matrix for B . in our work we do not depend for Hessian matrix.

Text numerical examples.

1. Consider the nonlinear programming problem with fuzzy function

$(FOP)min_{x\in R^2}\ \tilde{F}(x)$

$(\overline{f}^{\alpha}(x) + \underline{f}^{\alpha})(x) = e^{x_1-1} + e^{-x_2+1} + x_1^2 + x_2^2 - 2x_1x_2$

The initial condition $x^0 = (-20,15)\ , \epsilon = 0.00005$

2.Consider the nonlinear programming problem with fuzzy function

$$(FOP)min_{x\in R^2}\ \tilde{F}(x)$$

$(\overline{f}^{\alpha}(x) + \underline{f}^{\alpha})(x) = 6.025x_1^2 + 10.025x_2^2 + x_3^2$

The initial condition $x^0 = (1,1,2)\ , \epsilon = 0.00005$

3. Consider the nonlinear programming problem with fuzzy function

$$(FOP)min_{x\in R^2}\ \tilde{F}(x)$$

$(\overline{f}^{\alpha}(x) + \underline{f}^{\alpha})(x)$=$x_1^4 - 8x_1^3 + 25x_1^2 + 4x_2^2 - 4x_1x_2 - 32x_1 + 16$

The initial condition $x^0 = (1,1), \epsilon = 0.00005$

VI. CONCLUSION

If the determinant of the a approximation Hessian matrix for the fuzzy problem of high dimensions it is near zero, then he invers of Hessian matrix may be unavailable. We will found the numerical solution of fuzzy optimization problems. In this article we eliminated this defect by preserve the value of the determinate for the approximation Hessian matrix by modified the BFGS update, called $\vartheta - BFGS$.

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